THE
AMIR-UD-DAULA PUBLIC
LIBRARY
LUCKNOW

Class No... 510.92...

Book No... A31S...

दान द्वारा
Gifted by
राममोहन राय पुस्तकालय प्रतिष्ठान
RAJA RAMMOHUN ROY
LIBRARY FOUNDATION
K DD-34, SECTOR-I, SALT LAKE
CALCUTTA-700 064

THE BOOK

The present book has been written at the suggestion of the Indian National Science Academy, New Delhi and has also been financially supported by the Academy. It gives a popular account of the life and works of Ramanujan, the celebrated Indian Mathematician. The mathematical contributions of Ramanujan have been dealt with in a style so as to inspire an average university mathematics student to read more of it. A characteristic feature of the book that distinguishes it from others on Ramanujan, is that the author has tried to collect personally and very meticulously the details of personal anecdotes, life and style of functioning of the great genius from various possible sources which include personal interviews, visits to his native place and various libraries where information regarding Ramanujan was available in India.

The book is concluded by number of mathematical articles by experts who have worked and/or working on the various problems connected with Ramanujan's work. These add spice to the entire work.

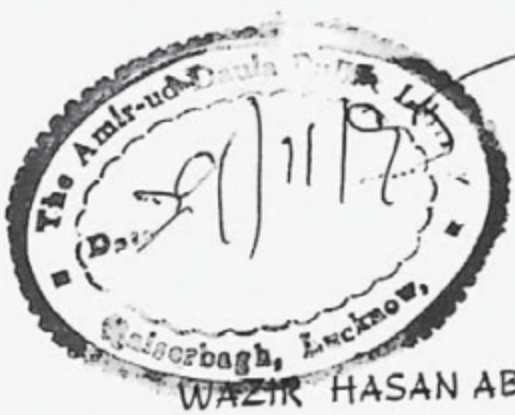
The book is expected to be useful to both, a general reader interested in knowing about Ramanujan and also to those who might be interested in deriving inspiration from his 'mystic' and 'ingenious' research work.

Toils and Triumphs of

SRINIVASA RAMANUJAN

The man and the mathematician

Toils and Triumphs of Srinivasa Ramanujan the Man and the Mathematician



WAZIR HASAN ABDI

NATIONAL

NATIONAL PUBLISHING HOUSE
CHAURA RASTA, JAIPUR-302 003

Library
R647
Class 510.92
Book No. A31 S

© Wazir Hasan

RAJA RAMMOHUN ROY
LIBRARY FOUNDATION

दान द्वारा
Gifted by
राजा राममोहन राय पुस्तकालय प्रतिष्ठान
RAJA RAMMOHUN ROY
LIBRARY FOUNDATION

BLOCK DD-34, SECTOR-I, SALT LAKE
CALCUTTA-700 064

First Published 1992

Printed in India at Rajendra Printers, S-2 Shopping Centre, Tilaknagar, Jaipur and Published by National Publishing House, Chaura Rasta, Jaipur-302003

Dedicated to the memory of
Sir Francis Spring
a friend of India, a friend of mathematics

A Word of Thanks

At the meeting of National Commission for Compilation of History of Sciences in India on April 19, 1986, the Chairman, Professor C.N.R. Rao remarked that it was a pity that our University students did not know about their own scientists of the recent past who had once again put India on the scientific map, and worse still, even teachers found it difficult to talk about these stalwarts to their students for half an hour so as not only to increase their knowledge but also to inspire them. He remarked that 1987 and 1988 were the Centenary years for Ramanujan and Raman respectively, so it would be in the fitness of things that monographs on their life and works were published by the Indian National Science Academy in 1987 and 1988 respectively. Someone suggested my name for preparing the monograph on Ramanujan, as I had written a short biography of Ramanujan in Hindi for school children more than two decades ago. At the meeting itself, I agreed to write a popular account of Ramanujan's life, indicating the broad lines of his mathematical contributions so as to inspire an above-average degree-student of any Indian University and whet his appetite for Mathematics. With this objective in mind, the monograph was divided into three sections— Section I gives an outline of Ramanujan's life, although mathematics also has been included at the appropriate places. Section II contains the contribution that Ramanujan made to the *Journal, Indian Mathematical Society* that he made before he left for Cambridge—a couple or so of papers which were published in 1915, were presumably submitted on the eve of his departure. The idea behind the management and choice was to show how

mathematical research can be conducted unaided so as to encourage the more enterprising student to attempt some similar problem himself. Finally, section III is a collection of survey articles by experts on various areas of Ramanujan's mathematical work.

In this task I was fortunate enough to get the co-operation and help of many seniors, colleagues, friends and associates, my former research scholars and young mathematicians. I am highly grateful to them because if they had not extended their ungrudging help, it would have been impossible for me to complete the task. First and foremost, my respectful thanks are due to Professor K. Venkatachaliengar (Bangalore), twice President of Indian Mathematical Society, who not only discussed the broad outline of the monograph for two days but also kindly lent me some valuable literature. He very kindly contributed an article also. Then my friend *(anna)* M.S. Rangachari of Ramanujan Institute arranged my interview with Thirumati Jankiammal and acted as the interpreter. He also gave me valuable information which was not available elsewhere. Balachandran provided me facility of the Ramanujan Institute Library, which houses the library of Indian Mathematical Society also. Srinivasa Rao (Institute of Mathematical Sciences, Madras) helped me with translation of relevant Tamil literature and gave some valuable suggestions. I had no idea, whatsoever of Nandy's article but Gratten—Guiness of Enfield (England) not only told me about it but also promptly sent (what he called) a (photocopy). S. Parameswaran (Trivandrum) besides contributing an article wrote to me pointing out certain problems and I found his monograph in Malyalam of great help. I am grateful to all these colleagues.

My former research scholar, K. Velukutty, a senior mathematician at Rama-Krishna Mission Vidyalaya College, Coimbatore acted as my guide for Erode and Namakkal. He was the interpreter between the head *Pujari* of Namagiri Temple, Shri Bhattacharya and me. Through him I met Sri Lakshminarayana, a mathematics teacher at the Islamia Madrasa School, Erode, who has done extensive research on Ramanujan. Much of the information about Erode, Namakkal and Kumbhakonam I owe to Velu (Velukutty). Sri R. Viswanathan, who is the present (30th)

My attempt has been to avoid serious errors but am I immune ? I will smilingly confess my incompetence before the elite of Indian Mathematics but will feel proud of my toils being rewarded if after reading these pages even a handful of students from grass—root Indian Universities are inspired to carry forward the torch lighted by Ramanujan.

Wazir Hasan Abdi
Hasan Manzil
89 H.IG. & E Sector,
Aliganj Extension
Lucknow–India
226020

Headmaster of Kumbhakonam Town Higher Secondary School and Secretary, 'Sri RamanujanCentenary Celebration Committee, was kind enough to provide the school's history. Another former research scholar of mine S.A.H. Rizvi was helpful in so many ways My thanks are due to Kamaladevi Singh (Head, Department of Mathematics and Astronomy Lucknow University) who allowed me reading facility at the Departmental Library. Finally I must thank the experts, Michael Hirschhorn, Arun Varma, M.A. Pathan, besides Venkatachaliengar. Parameswaran and R.P. Agarwal who contributed their valuable articles on various aspects of Ramanujan's work for Section III. Professors Andrews. Berndt and Ramchandra sent their suggestions for which I am thankful. My thanks will also go to two youngman Bhaskaramurthy of Madras and Amit Saxena of Lucknow for help with translation and fine typing respectively.

My major information in Sections I and II is based upon the monographs of Berndt. Ranganathan and Srinivasan and various issues of *Journal, Indian Mathematical Society* besides the *Notebooks*. It was inevitable that free use of the above source material had to be made.

Lastly, I must thank the Academy for entrusting the job to me and providing financial support. My special "Thank you" is due to my friend A.K. Bag, the Assistant Executive Secretary (History of Science) for his patience as he was assuring everyone that I would not be a defaulter. I will fail in my duty if I do not express my indebtness to Anand Goyal and Devendra Malik without whose efforts the monograph could not see the light of the day.

Let me add that inspite of the maximum care taken by the printers, some errors still were left undetected. Accordingly an Erata has been appended. I would request the readers to let me know about any error if they find one, so that it may be rectified in the next edition.

I close with expressing gratitude to my teacher R.P. Agarwal, who supported me throughout with his suggestions and advice.

Contents

I. (THE MAN)

	A Word of Thanks	(i)
1.	Introduction	1
2	Little Master of Wet Skull	2
3.	No mangoes for no people.	9
4.	Really amazing. But sorry, can't help	22
5.	Spring brings new life	30
6.	Amman ! Seven Seas ?	46
7.	I came, I joined, I Conquered	53
8.	Sick and tired ? Not me	66
9.	Goodness Gracious ! Black diamond	78
10.	Where is she ?	87
11.	Vanakkam Chinna !	94
12.	My dear bull had big eyes	99
13.	Puffed Month ? Never	105
14.	Treasure in scribblings	113
15.	When the Sun goes down	121

II. (PUBLICATION BEFORE CAMBRIDGE TRIP)

1.	Indian Mathematical Society and Ramanujan	123
2.	Some Properties of Bernoulli's numbers	129

3. On Question 330 of Professor Sanjana 151
4. Note on a set of simultaneous equations 156
5. Irregular numbers 158
6. Squaring the circle 161
7. On the integral $\int_0^x \frac{\tan^{-1} t}{t}\,dt$ 163
8. On the number of divisors of a number 169
9. On the sum of the square roots of the first n natural numbers 173
10. On the product $\prod_{n=0}^{\infty}\left[1+\left(\frac{x}{a+nd}\right)^3\right]$ 178

III. (SURVEY OF WORK)

1. Ramanujan's Contributions to the theory of elliptic functions—K. Venkatachalingar 182
2. Ramanujan and the additive theory of numbers —S. Parameswaran 207
3. Identities of Rogers—Ramanujan Type—A. Varma 219
4. Ramanujan's Contribution to Continued fractions —Michael D. Hischhorn 236
5. Ramanujan's work on hypergeometric series and integrals—M.A. Patan 247
6. Ramanujan and Theta Functions—R.P. Agarwal 265
ERRATA 285

1

Introduction

Who was Ramanujan ?

A son of India, born in 1887, struggling for 32 years 126 days left the world, leaving some Notebooks behind.

What is Ramanujan ?

A college drop out ? A petty job-seeker ? An intellectual fraud ? A genius ? A wonder from the East ? An underdog in the West ? A being with divine power ? Deity's medium for mathematics ? A gift from heaven ? A gift from Britain to India ? India's gift to the world ? A phoney mathematician with unparalleld insight ? A Cambridgeman's alter-ego ? A religious Hindu ? An agnostic ? A psychological freak ? A Freudian Specimen ? An anachronistic computationist ? Creator of mind-boggling mathematics ?

For an answer, read the following pages, weigh' think and study further.

2

The Little Master of Wet Skull

It was a cool evening of the twenty-second day of December, a century ago (1887). The Moon was rising high above the firmament of Wet Skull when a little baby boy was born. The mother, Komalatammal was happy and her mother Rangammal was very happy as her prayers had been answered at last. The baby was lovingly called the "Little Master"—Chinnāswāmi.

Rangammal lived in Erode. Her husband, Nārayan Iyengar, a high caste Brahmin, was *amin* (a petty official) at the local *Munsif's* (subordinate judge's) court. As was the custom then, the daughter had come all the way to deliver her first baby at the parents' house. Erode, though not very big yet was important. The name of the town "Wet Skull" in Tamil is derived from the "Skull"of Brahma found in one of the hands Ārdra Kapaleśwāra, the presiding deity of Siva temple. The legend is that in the days of yore, Dakshaprajāpati, who had given his daughter, Dakshāyani in marriage to Siva, once performed a *yagyam*(sacrificial ceremony) to which he invited all the gods except Siva. Dakshāyani took her husband's permission and went to witness the sacrifice. But being unwelcome there, she felt insulted and threw herself into the fire burning herself to ashes. An enraged Siva then rushed to the sacrificial spot and killed everyone present there. He did not spare even god Brahma who was there to accept the offering. Siva tore off one of his five heads and returned with the skull. This is how the place got its name Erode.

Today Erode is a municipal town 99 kilometers to the north-east of Coimbatore. The District Gazetteer of Coimbatore tells us, during Haider Ali's region (1761–1782) it was a very flourishing town with, 3,000 houses and a population of 15,000. But owing to successive wars and invasions by the East India Company, it became almost utterly deserted and ruined. However as soon as peace was restored people returned and settled in comparatively fertile places and soon it began to grow with a population of over 2,000 living in 400 houses. There is still a fort, called *Koṭṭai* to remind of the past ravages.

Even nature has not been very kind to Wet Skull. Rainfall is the lowest of the rainfall in the entire Tamil Nadu. Even the river Kaveri which is joined by a drainage channel Koranganpallam, does not provide any irrigation. until it passes Erode its bed is rocky and its flow very rapid. As a result scarcities have very often and famines times and again visited the area. It has been recorded that during the nineteenth century famines have occured in 1804 to 1806, 1808, 1812, 1823, 1831, 1832, 1834, 1836, 1861 and 1866 prior to the Great Famine of 1876–1878. Although the worst was over by the end of 1878, the sequel had to be faced for a long time afterwards. The relief camps became hospitals, cholera killed thousands and dysentry, fever, and anaemia tens of thousands. It has officially been estimated that in Coimbatore district 1,97,188 people and 2,06,859 cattle died during the famine and Erode also bore the brunt.

Deeply struck in the mire of natural calamities, abject poverty all round and inadequacy of material conditions, people had no alternative but to turn towards their particular deities who were invoked to fulfil the poor devotee's desire for better means of livelihood, good health and children, rains and flourishing harvest. That is why Erode itself was dotted with numerous temples dedicated to Siva or Vishnu. But the most important shrine for *Srīvaishnavites* (devotees of Vishnu) was at Namakkal situated 58 kilometers to the south-east of Erode. It lies at the foot of rounded man of rock whose colour resembles that of *nāmam*, the thin lines of sandalwood paste drawn on the forchead of srīvaishnavities. This gave the town its name Namakkal.

At the base of the rock stands a vaishnavite temple dedicated to Narasimha the fourth *avatār* (incarnation) of Vishnu and his consort Lakshmi, locally known Nāmagiriamman or Nāmagirithayar. Inside the sanctum of the temple there are the idols of Vāman, Varāha and Narasimha avatars, Vishnu and Vaikuntha cut in monolithic cave *mandapam* (altar) of the principal deity. According to the Salem District Gazetteer, they are perfect works of art of the Pallava period (7th–8th centuries) not found even in Mahābalipuram. Facing the gateway of the temple stands a colossal idol of Hanuman whose eyes seem to be glued to the feet of his master Narasimha. Devotees flock to the temple in large numbers especially when someone in the family is suffering from a serious ailment which is attributed to the patient having been possessed by a devil. It has been noted in the same Gazetteer that on Tuesdays and Fridays the hall in front of the Goddess is filled with the shrieks and convulsions of women until sprinkling of sacred water over their heads silences them. Rangammal used to come all the way from Erode to worship Nāmagirithāyar very frequently. Having a *darshanam*(sight) of the idol she would go into a trance and would feel that the goddess was speaking through her. On one of such occassions, she had a vision of the goddess who gave her boon that the daughter Komalatammal would have a son and Nāmagirithāyar herself would dwell on his tongue. And soon afterwards Chinnāswāmi came into this world.

About two weeks after his birth Chinnāswāmi was given his official name–Ramanujan–at a *nāmkaran* (naming) ceremony after the great thinker, religious reformer and philosopher–Ramanuja–(1017–1137) who had given impetus to the spread of srivaishnavism. He also believed in Vedānta or Uttar Mimāmsa darshan but disagreed with Sankara's doctrines regarding existence, *māyā*, causality and relation of God with Self. To him the material world was not an illusion, in fact all knowledge was real. In spreading his teachings he broke many taboos of rigid caste systems and admitted Jainas, Buddhists, Śudras and even Untouchables to his fold. He himself was the pupil of a Śūdra—Kānchipūrna—and used to spend a long time after bath in the hut of an untouchable friend of his. And as a historian of Indian philosophy, Surendra-

nath Dasgupta tells us "in the Āṛvars (vaishnavite saints) the srivaishnava movement was primarily a religious movement of mystic and intoxicating love of God and self-surrender to Him. In the days of Ramanuja it became intellectualised for some time, but it slowly relapsed into the religious position."

With her desire for a baby boy fulfilled by the grace of Nāmagirithāyar, a happy Komalatammal returned to Kumbhakonam with Chinnaswami in her arms where her husband lived. Kumbhakonam is a town in Tanjavoor district, on the banks of Kavery 305 kilometers from Madras towards south. It is also called the City of Temples as it has more than a hundred temples dedicated to three deities-Brahma, Vishnu and Siva. *Kumbha* means pitcher in Sanskrit and the city draws its name from the legend that when both gods and demons churned the ocean to take possession of the pitcher of *amṛita* (elixir) some demon ran away with the pot. Gods tried to snatch it and in this melee, portions of the pitcher fell on various places in India. Near the present Kumbhakonam, a portion of the pitcher fell on a spot which is a ten acre expanse of water known as *Māhāmāhākulam*. When Jupiter completes one round about the Sun in twelve years, there is a big religious festival and devotees throng in hundreds of thousands to have a holy dip in the Mahāmahākulam. The festival is called the *Mahāmahāvizha* similar to the Great Kumbha Mela held in Allahabad at the confluence of Ganga and Yamuna.

In this holy city of temples, the holiest temple was the Sārangpāni Temple which was hallowed by the hymns of Āṛvars, the srivaishnavite saint-poets. It was in a nearby street – Sarangpani Sannidhi–where Chinnaswamis parents lived in a small house, His father, Srinivas Iyengar eked out a living by working as a *gumāshta* (shop assistant) at a cloth merchant's shop getting a paltry sum of Rupees 20/– per month. In order to make both ends meet, the mother, Komalatammal, joined a group of women *bhajanīkās* (singers of hymns) who used to sing at Sarangapani. The leader of the choir Venkūammal used to collect sixty to hundred rupees per month from the devotees which she distributed amongst the *bhajanīkās*. Thus Komalatammal would get 5–10 rupees a month. Besides, she

would take two boarders occasionally charging Rs. 10/- per month from each. Even with all this the poor women could not keep the wolf from the door. Then the family grew – she had two more sons. Very often the parents would give scanty food to the children and themselves go to bed without any food. Some better off neighbours would occasionally help the family with some rice and sometimes feed the children. Once when mother told him that there was no rice in the house, little Chinnaswami drank some water and was later found sleeping on the steps of the temple.

When Chinnaswami had crossed three, he had his first lesson at a simple ceremony called *aksharābhyāsam* (learning the alphabet) and then he was sent to a *pial*, to learn reading and writing Tamil, elementry rules of arithmetic and some *ślokas* (verses) in Sanskrit used in daily prayers. A *pial* was a conventional type of elementary school set up at a poor teacher's house where the children would sit on two platforms on either side of the path leading to the door, with the teacher sitting in the middle in front of his door. Here Chinnaswami spent two years. His classmates used to tease him but he would be oblivious of their childish pranks and remain engrossed in his work. When he was in his fifth year he had his *upānayan* (initiation ceremony) and from then on it became compulsory for him to wear the sacred thread on the body and a tuft of hair at the back of his head.

After leaving the *pial* if little Chinnaswami had joined the stream of classical education, he would have had to study Sanskrit, Grammar, Philosophy and with the natural bent of mind he would have had to spend his lifetime in learning astronomy and the works of great Indian mathematicians like Āryabhaṭṭa, Bhāskara etc and if he had access to sixteenth century classic *Tantrasamgraham* dealing with infinite series he might have been tempted to develop the ideas contained therein. But he was born at a time when the edifice that had earlier produced such mathematical giants had been shattered as a result of continuous efforts made to impose western civilisation. As has been noted by various historians of education "the period 1854–1902 is marked by westernisation of education and neglect of indigenous schools leading to their complete disappearance by 1902. English education had

come to be looked upon as a passport to clerical and professional employment."

It may be recalled that when in the first decade of the nineteenth century the officials of East India Company were hesitating about introducing English in their colleges for fear of provoking religious repurcussions, they got a rebuke. In their despatch of 14th February 1824, the Board of Directors pointed out that "the great end should not have been to teach Hindu learning or Mahomedan learning, but useful learning" and it was a deplorable policy "to teach a great deal of what was frivolous, not a little of what was purely mischievous and a smaller remainder indeed, in which utility was in any way concerned." This line of thinking was further confirmed by the despatch of 1854 which affirmed "We have always been of opinion that the spread of education in India will produce a greater efficiency in all branches by enabling you to obtain the services of intelligent and trustworthy persons in every department of Government; and on the other hand, we believe that the numerous vacancies of different kinds which have constantly to be filled up may afford a great stimulus to education." After the reigns of government were directly taken ever by the British sovereign from the Company the policy of westernised education for providing petty jobs was continued and later with the establishment of Universities, this type of education opened the door for some junior positions in the governmental machinery also.

No wonder then that little Chinnaswami moved in 1894 from the humble *pial* to a slightly less humble Kumbhakonam Kangeyan Primary School. His name in the school register was entered as S. Ramanujan, son of Srinivasa Iyengar. Here his interest in Arithmetic increased and when his classmates would take ten minutes to do a sum, he would hardly need two minutes. The headmaster M. Mahadeva Rao was a strict disciplinarian, but took special care of good students. At the end of four years Ramanujan took the final examination in November 1897. All the boys taking the examination had to write the answers on slates and marks were awarded on the spot. Ramanujan passed in the compulsory subjects : Tamil language 1. reading, recitation and

grammar, 2. writing and spelling 3. Arithmetic, besides the optional subjects 4. English and 5. Geography. He was placed in the first class,—in fact he was first not only in his school or Kumbhakonam but in the entire Tanjāvoor district. But his self-confidence in Arithmetic was so strong that when a classmate secured 43 out of 45 and he got only 42, Ramanujan wept bitterly. The certificate awarded in the name of Government of Madras and signed by J. Daniel, Additional Chairman, Board of Examiners Primary Examination, Tanjor (Tanjāvoor) district on 9th August 1898, stated "S. Ramanujan is hereby declared to have qualified to admission in Public Service in accordance with the provisions of the Public Service Notification."

It was open for the son of a poor *gumāshta* and *bhajanīkā* to try for a job and provide a helping hand to the parents, but Ramanujan chose to continue his education and face a life of struggle.

3

No Mangoes For No People

Ever heard about a school with a policeman as the Headmaster. ? Yes, at least there is one—the Town High School at Kumbhakonam !

The school was established on April 14, 1864. The first Headmaster was a Head Constable of police and a Briton at that, Porter by name. The choice was due neither to any law and order problem concerning the students nor any prejudice against employment of Indians to the post. In fact the number of students in the first batch was very small and only those pupils had joined the school who were really keen at studies. The main reasons were two : in those early years of English education, Indians well versed in subjects prescribed were not easy to find and above all teaching of English was the most important part of education, So it was believed that English should be taught properly only by someone whose mother-tongue was English. It was this very Town High School where Ramanujan was admitted to the first form in January 1898. During the thirty four years since its foundation, the school developed and had mostly Indians on its teaching staff; Krishnaswami Iyer was the Headmaster and Ganapathi Subbier the Mathematics teacher. As Ramanujan had secured the top position at his Primary School Examination in the entire Tanjavoor district, he was awarded half-freeship in tuition fees, and he started his studies more seriously not confining himself only to the lessons in the classroom or the prescri-

bed text books. He used to ponder on questions which not many of his teachers-let alone the classmates, found easy to answer. A quiet and meditative boy, who was not yet eleven, came to be respected by the students and teachers alike. He was always ready to help his class fellows in the solution of mathematical problems. Many of them would go to this house seeking help but as his parents would not allow them to came in, he would stand at the window and discuss with them.

By the time he reached the second form (equivalent to the present sixth class) at the age of twelve, he had acquired so much interest in Mathematics, that he wanted to know what its highest principle (or truth) was. Some students of higher classes told him that it was the Theorem of Pythagoras—and for some Computation of Stock and Shares was the highest principle which a mathematician would aspire to learn. Obviously their imagination could not fly beyond the school geometry and arithmetic. But Ramanujan could not be satisfied with these simple answers and by the time he was in his thirteenth year his critical faculty had developed to such an extent that he would not take even the teacher's word without being convinced. The teacher was one day explaining the Rules of Division to the students of third form. He said that if a quantity is divided by itself the quotient is one, and as an example said that if 4 mangoes are divided between 4 men equally, every man will get 1 mango. Ramanujan stood up and asked if there are no mangoes and no people how will each person get one mango. In fact zero divided by zero can be any number. It was obvious to Ramanujan that $\frac{O \times m}{O \times n} = \frac{O}{O}$ where m and n are any numbers, but it is not equal to 1—it could be anything but the poor teacher found the argument too hard to digest.

The style of work for Ramanujan, even at that early age was completely different from that of others. His classmates would study their courses in order to attain good marks or at least pass the examination as more students would fail leading to their dropout. It has been noted in the *Report of University Education Commission* (Dec. 1948—Aug. 1949) that in 1882 at the entrance

(High School) examination of the three Universities—Calcutta, Bombay and Madras as many as 7429 candidates appeared out of which 4,651 (62.61%) failed, only, 2,778(37.39%) could scrap through. But Ramanujan was keen on learning and studying what he liked best and as long as he was at school, examination would take care of itself. He used to borrow whatever books of Mathematics were available at the School library and when his hunger was not satisfied he would request some neighbour to borrow more advanced books from the College library. It was through one of such neighbours—a student of B.A. —that in the fourth form he got access to *Trigonometry* Part II by S.L. loney and immediately settled down to solve the problems unaided. This book deals with separation of real and imaginary parts, Demoivre's Theorem, terminating and non-terminating sums and products of triganometric functions, infinite series for sine and cosine and logarithmic functions. This book, or its adapted version is still prescribed for B.A.,B.Sc. 1st year students in many Indian Universities who find the problems contained therein difficult to solve. But here was a lad who had still five years to reach that class, who had no difficulty—in fact he would skip many steps in the solution, but arrive at the correct result. He even worked out the infinite series formulas for sine and cosine all by himself when he was in the fifth form, but later on when he learnt that Euler had already established them long ago, he concealed them in some corner of his house. Many students of B.A. would come to him without hesitation with their difficulties while some of his senior schoolmates were jealous of him. While he was in the fourth form a student of the final class (two years senior to him) sent a challenge to Ramanujan if he could solve the simultaneous equation $\sqrt{x+y}=7, \sqrt{y+x}=11$. Within half a minute Ramanujan gave a solution $x=9$, $y=4$. It may be noted that this system can be reduced to an equation of fourth degree in one variable with constant real co-efficients which would not be easy to solve. It is surprising how Ramanujan arrived at a solution in half a minute. But that was the style that he was developing; he would arrive at results without much explanation or applying the stereotyped techniques.

In the school, his mathematical skill had matured to such an extent that his teacher Ganapathy Subbier entrusted him with the difficult job of preparing the school time table for the higher classes where in the sixth form itself there were six sections with about 40 students in each. Time table making is a branch of Operation Research which is now treated as a branch of Higher Mathematics requiring ingenuity and analytic'skill-and here was a young boy coming from a poor family who was able to do it without any experience or training.

In the final year, 1903, a small event determined the entire course of his life. A friend borrowed for him from the college library a copy of a handbook *A Synopsis of Elementary Results in Pure Mathematics containing propositions, formulae and methods of analysis with abridged demonstrations* by G. S. Carr. As the title itself indicates the *Synopsis* was no mathematical treatise of very high standard. It was just like many cheap text books of today current in India where mathematical principles and methods are given in the form of questions set at various Universities' examinations and their answers. It contained enunciation of 6165 problems mostly on calculus and geometry and very little complex analysis or elliptic function theory. But Ramanujan treated its contents as pegs to hang his mathematical thoughts upon. He would take a problem, work it out and jot down the results in his Note-Books. The first Note-Book contained many of the formulae that he obtained while still at school. The earliest attempt was on construction of magic squares which is contained in Chapter 1. It covers only 3 pages of Note-Book I and later developed and further elaborated in Note-Book II covering 8 pages. Recently a detailed analysis of these results has been given by Bruce C. Brandt in his *Ramanujan's Note Books Part I'*. It may be mentioned here that Magic Squares, according to Kripa Shankar Shukla, were known in India as charms from the Ist Century A. D. and references to them are found in tantaric literature. Construction of magic Squares involves ingenuity and hence it has aroused interest of some eminent mathematicians in the past. Nārayan of fourteenth century has devoted one full chapter on construction of magic squares, oblongs, *vitāns*(canopy), *mandapas* (altars), *padamas* (lotus), *dvādashakara* (twelve hands) and

also magic triangles, crosses etc. in his *Ganita Kaumudi*. Shukla has also noted that in Persian and Arabic script, magic squares prepared by Indian mathematicians were as large as to have 100 cells in a row or column. But obviously, Ramanujan could not have any access to that literature so as to carry forward the Indian tradition.

Not finding much scope in Magic Squares based on Carr's *Synopsis*, he turned to geometry and worked out a few formulae, including a Theorem' entitled *Squaring the Circle*, which has been 'solved' in Chapter VI of Note book I. Later he described the steps in more detail on pages 220–221 (entry 20 (i) in his Note Book II and added a note that" the side of the required square is 1/100 th of an inch greater than the true length if the given circle is 14 square miles in area. He also added two corollaries :

Cor. 1 One of the mean proportionals between a side of an equilateral triangle inscribed in the circle and the length PS in only less by 3000th part of it than the true length.

Cor. 2. The approximate length got by assuming

$\pi = \sqrt[4]{97\frac{1}{2} - \frac{1}{11}}$ is 1/100th of an inch less than the true length if the circle is a million square miles in area.

Many years later he explained the steps further and gave the proof also which was published in *Journal of Indian Mathematical Society* V. 1913 p. 182. In the published paper he took the value of π as 355/113 but did not use the other value which he mentioned in Cor. 2 One wonders where he came across the two values – 335/113 is, obviously, known as the Chinese value which was used by Narayan in 1426, but what was the source of the other value. The problem of squaring the circle or circling a square, in fact, boils dowh to finding better and better approximation of the transcendental number π which requires more sophisticated mathematical tools than what were available to Ramanujan in the first decade of the present century. However he was able to publish a 23 page paper in the *Quarterly Journal of Mathematics*, XLV, 1914

pp. 350 entitled *Modular equations and approximations to* π where very ingenious methods have been followed.

While he was busy in his 'discoveries' based on Carr's *Synopsis*, he did not turn out to be a bad examinee as long as he was at Town High School. Every year he got good grades and was awarded prizes "as a reward of merit and an incentive to further improvement". At the Annual prize distribution when he was in the Final class (Sixth form) Krishnaswami Iyer, the Headmaster, very proudly presented Ramanujan to the president of the function saying" here is a student who, my Mathematics assistant says, has displayed in his answers a remarkable ability and deserved more marks than the maximum itself." The prizes included :

1. *Wallace and Bruce. Heros of Scotland* by Mary Cochrane,
2. *Poems of England. A Selection of English Patriotic Poetry with Notes* by Hereford B. George and Arthur Sidgwick,
3. *Cranford* by Gaskell,
4. *Critical and Historical Essays* contributed to the *Edinburgh Review* by Lord Maculay,
   and for proficiency in Mathematics'
5. *The Poetical works of Willam Wordsworth* with introduction and notes by Thomas Butchinson.

Whatever might have been the idea behind the choice of books as prize, Ramanujan could become neither a hero of Scotland nor a patriotic Englishman nor an Indian Wordsworth. He did pass in December 1903 the Final school Examination (Matriculation) of Madras University in the First Division only to fail in English one year later.

A few weeks after his sixteenth birthday, Ramanujan entered the Kumbhakonam Government College as a student of First Arts (F. A.) class, equivalent to the present day 1st year Intermediate or class XI. The College was established in, 1854, when upto

1857 there were only 27 colleges throughout British India, and Madras presidency which spread over all the present South Indian States Tamil Nadu, Andhra, Karnataka and Kerala– had its share of four colleges– Kumbhakonam being one of them. But as has been observed by a historian of education" during those days a very large number of candidates who passed the Matriculation sought admission to universities mainly due to material advantages that were then attainable by holders of university degree". Consequently by 1882, the number of colleges increased to 72 (a rise of 166.66%) and in Madras Presidency to 25 (an increase of 525%). The universities were only examining bodies, teaching was the responsibility of affiliated colleges. Almost all the colleges were run either by the Government or foreign Christian missionary societies. However, during that period five colleges were established under Indian management also Mahomedan Anglo-Oriental College of Aligarh and Canning College of Lucknow which later developed into Aligarh Muslim University and Lucknow University respectively both located in the then North-Western Province. The remaining three were in Madras Presidency : Pachaiyappa College arose from a school established at Madras in 1842 from the funds derived from a bequest for "pious purposes" by a wealthy Indian Pachaiyappa Mudaliar, the two other colleges being situated at Vishakhapattanam and Tinnevelly.

On admission to the College, Ramanujan wrote a competitive test for Junior Subramanian Scholarship. He got 100% marks in Mathematics but 3 marks less than the required minimum in English Composition. However on the advice of a teacher he took the supplementary test and got the scholarship, which relieved him of financial worry to some extent. The course for F. A. Ist year consisted of (1) English, (2) Mathematics, (3) Greek and Roman History, (4) Physiology and (5) a second Language–classical or vernacular. For that matter Madras was the only university that allowed the option of an Indian language–say Tamil as a second language. Ramanujan preferred to offer Sanskrit. He was a hard working student, but none of the subjects other than Mathematics had any attraction for him. Even during lectures of other subjects, he would be busy with his Mathematics' mainly

outside course work. At first even his Mathematics teachers were not able to understand this strange student. T. N. Venkatarama Ayyar poohpoohed him, K. S Patrachariar evinced some interest and P. V. Seshu Iyer was indifferent, but it was Seshu Iyer himself who soon realised the potentialities of Ramanujan. During lectures, he would ask the students to come to the blackboard and solve some connected problem but he would let Ramanujan alone, busy with his researches in studying, solving and extending the problems contained in Carr's *Synopsis*. Even at that stage Ramanujan tried and found a number of interesting results in Finite and Infinite Series, Bernoulli Numbers without knowing that similar results had already been established much earlier elsewhere. He also learnt Differentiation and Integration by himself and tried some series known in literature as Hypergeometric Series.

He used to jot down whatever result he obtained in some note book which he kept with him all the time. Sometimes he gave the detailed 'proof' and occasionally omitted any explanation, as he treated the results as 'aids to memory.' Amongst the earliest results are the Theorems contained in the First Note book, For example in Entry 1 of the Second chapter (page 4) using the simple identity

$$\frac{1}{(2n)^3-2n} = \frac{1}{2(2n-1)} - \frac{1}{2n} + \frac{1}{2(2n+1)},$$

he proves that

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots\ldots\ldots + \frac{1}{2n}$$

$$= \frac{n}{2n+1} + \frac{1}{2^3-2} + \frac{1}{4^3-4} + \frac{1}{6^3-6} + \ldots\ldots\ldots + \frac{1}{(2n)^3-2n}$$

and infers that

$$2\log_e 2 = 1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \frac{2}{6^3-6} + \frac{2}{8^3-8} + \ldots + \text{ad inf.}$$

Proceeding on the same lines in Entry 6 (pages 7-8) he shows that if $K_r = 3^r(n+\frac{1}{2}) - \frac{1}{2}$ then

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{K_r} =$$

$$= \left\{ 1 + \frac{2}{3^3-3} + \frac{2}{6^3-6} + \frac{3}{9^3-9} + \ldots + \frac{2}{(3n)^3-3n} \right\}$$

$$+ (r-1) \left\{ \frac{2}{(3K_0+3)^3-(3K_1+3)} + \frac{2}{(3K_0+6)^3-(3K_0+6)} + \ldots + \frac{2}{(3K_1)^3-3K_1} \right\}$$

$$+ (r-2) \left\{ \frac{2}{(3K_1+3)^3-(3K_1+3)} + \frac{2}{(3K_1+6)^3-(3K_1+6)} + \ldots + \frac{2}{(3K_2)^3-3K_2} \right\}$$

+...to r terms.

As a corollary, he obtains the result

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{\frac{3^n-1}{2}}$$

$$= n + (n-1) \left\{ \frac{2}{3^3-3} \right\} + (n-2) \left\{ \frac{2}{6^3-6} + \frac{2}{9^3-9} + \frac{2}{12^3-12} \right\} +$$

$$+ (n-3) \left\{ \frac{2}{15^3-15} + \frac{2}{18^3-18} + \ldots + \frac{2}{39^3-39} \right\} + \ldots \text{etc to n terms.}$$

At the end he adds a Note : Entry 6 and the Corollary are very useful in finding the approximate value of the reciprocals of the first n natural numbers whether n is small or very great without knowing logarithms, differential and integral calculus. It must be remembered that when a_1, a_2, a_3 etc. are in A.P., the approximate value of $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_1} = \frac{2n}{a_1+a_n}$.

The result is followed by Example 1 :

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12}$$

$$+\frac{1}{13} = 3+\frac{1}{6}+\frac{1}{105}+\frac{1}{360}+\frac{1}{858}$$

which ends with an exercise : show that

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{100} = 7\frac{1}{2} \text{ very nearly.}$$

It may be noted that during the first year of his at college life he established hundreds of such results which indicate how a simple identity becomes a tool in the hands of an ingenious student of Mathematics in creating beautiful results just as a few simple lines drawn by Picasso produce an effect which leaves one wonder-struck.

While busy in 'spinning' result after result, he suddenly fell ill with some mental problem, but that could not halt his progress in mathematical discoveries or rediscoveries. Any worry for the Annual Promotion Examination even could not become a hurdle. He was simply ignoring his course-work in all subjects other than Mathematics inspite of repeated warnings and advice by his teachers-especially Seshu Iyer. So the result was obvious. He failed at the Examination, no promotion to Senior F.A. class, no financial support, end of Subramanian Scholarship, All the teachers were sorry for him but what could they do. Poor mother Komalatammal appealed to the Principal for mercy, but Nagaraja Rao expressed helplessness, his hands were tied by the strict rules of promotion.

A dejected Ramanujan ran away from home. He was told by some friends that in Vishakhapattanam, a town about a thousand kilometers to the north of Madras, there was some possibility for him to find a patron who could help him in continuing his studies and mathematical activity. So away he went taking with him his only possession-the Note books But who come to the rescue of a poor lad of seventeen shabbily dressed claiming to have produced mathematical results which were beyond comprehension of the rich or even practising mathematicians. So he had no alternative but to return home empty-handed.

But as happens in such cases, every student has some teacher or the other who knows the strong and weak points of his pupil. For Ramanujan such a teacher was Seshu Iyer. It was he through whose advice Ramanujan earlier cleared the Supplementary Examination for Subramanian Scholarship. He also used to go through the results obtained by Ramanujan and now it was none other than Seshu Iyer who consoled him and advised him to leave Kumbhakonam, migrate to Madras, where he could get some financial support by giving tuitions and continue his studies at Pachaiyappa College. So away he went to Madras and met Ramanujachariar who taught Mathematics at Pachaiyappa. The teacher had a good look at the Note Books, but could not believe that a boy who could not pass even the Junior F.A. Examination of Kumbhakonam College was the author of such mathematical results. So he gave him a problem thinking it to be exceedingly difficult but Ramanujan came out with a solution in less than a minute. From then on Ramanujachariar became an admirer of Ramanujan. On his recommendation Principal J.A. Yates granted half-freeship in fees and promised to give the rest from his own pocket to this talented youngman.

Ramanujan became a regular student once again, but there was no change in his habits of study. He had no love for English or Greek and Roman History : they were all alien to his background. And he despised Physiology and would sometimes simply laugh at what was taught in the class. His one and only one love was Mathematics and Carr's *Synopsis* continued to be his guide. Financial stringency and poverty had followed him to Madras also. His parents were not able to help him, whatever money he was able to earn was through giving tuitions. How could he afford nutritious food without the mother's loving care. He would take his meals at some way-side eating house where only the poor used to rush for a scanty meal. The result of all this was that within four months of joining Pachaiyappa, he fell seriously ill and had to discontinue his studies. Again the same story-at any examination, he continued to secure 100% in Mathematics, never more than 25% in English, 15-20% in Greek and Roman History and a measly 10% in Physiology. Back home in Sarangpani Street of Kumbhakonam, he found a very cold

welcome. Mother was silent. Father, who was generally indifferent towards his children's welfare as he had no energy left for them after being sapped working the whole day at the cloth shop, was very disappointed as the son who had secured a First Class at Mat riculation had dashed all hopes of becoming a helping hand and ameliorating the financial condition of the semistarving family. He said nothing to Ramanujan, but told Komalatammal "Your chinna is worthless, good for nothing-only you have pampered him. He will never do anything in life". Poor Ramanujan once again did appear privately in 1907 at the F.A. Examination and again failed.

But was he really worthless? After all, his Mathematics was of a level beyond comprehension of many teachers. But the other subjects were alien to his culture. He loved and was keen on what was native to him - literature, drama and the like. While discussing mathematical problems he used to quote proverbs and allegories from folk tales of Tamil Nadu. He would, occasionally, be found humming the songs sung by women while watering paddy fields. He enjoyed watching *therukkattu* (street plays) and would never miss a chance of watching them if he heard about them. These plays enacted by local artistes depicted some story from the Indian Epics and Puranas like *Ramanatakam* (the story of Ram), *Krishnalīlā*, *Prahlād-Harnākashyap*, scenes from *Mahābhāratha* and the life of *Harīshchandra* who sacrificed his all for the sake of truth. Incidently this last drama, played in another Indian language,-Gujarati, influenced Mahatama Gandhi as a young boy in the formation of his character. Chinnaswami was so keen on these plays that one night he walked about 5 kilometers all alone in the night to a village Thepperamannallur passing through a cremation ground where a corpse was burning. Next morning when asked why he was not afraid, Chinnaswami philosophised saying "death comes after life and life comes after death".

Who knows if a young man with such artistic and philosophical tastes compounded with love for Mathematics could have shone as a brilliant examinee also if the syllabi were different. But then who is to blame for Ramanujan's sad plight at College. He himself? Possibly nobody? or someone else? Or was it the system

of education itself? May be, Lord Curzon has a reply, whose biographer, Raleigh, notes that the Viceroy, Lord Curzon called a conference of education officers at Shimla in September 1901, "where with his characteristic thoroughness he reviewed the whole field of education. While admitting that the successes of the British in imparting English education to India had been greater than their mistakes and blunder he deprecated *the too slavish an imitation of English models*. He thought that the existing system of examination had been pushed to an unhealthy excess and that cramming had been elevated to the level of a high art and that teachers cared more for results than for actual education of the taught. The colleges were mere collections of lecture rooms and laboratories, flung far and wide over provinces, bound to each other by no tie of common feeling and affiliated to examining bodies called Universities for which these institutions had no filial revence."

And Ramanujan could not come out unseathed through this very grinding mill.

4

Really Amazing, But Sorry Can't Help

What does a youngman, who is unwelcome at home on being thrown out of college, do? Does he sit down and brood? Or take to antisocial activities? Poor Ramanujan, who was once a Chinnaswami was in such a predicamnet. He did sit down but not brood. As long as Carr's *Synopsis* was with him and the will to produce mathematics was strong, he could not remain idle, he continued to find newer and newer results without knowing that many of his results had already been anticipated long ago. And then, the only means of earning a few rupees open in such cases was to give tuitions. His reputation as a master of mathematics had transcended the four walls of schools and colleges. He was the most sought of tutor, although he would not get more than 5 to 7 rupees tutoring a student daily two hours for one full month. But at least this made him feel that he was not only 'producing mathematics' but also 'earning money'.

But how long can a mother see her son, worthy or unworthy in such a sad plight? Komalatammal was working out her own solution. She felt that if a lovely girl was found for him, Chinna may possibly become a worldly wise man after marriage and do something serious for his livelihood as his love may shift from mathematics to the bride or at least be equally shared. One day in 1908 she met her friend Ranganayakiammal whose husband

Rangaswami was a *mirasdar* (rich landowner) of Rajendram,a village near Marudur railway station. There she saw Janaki. The eight year old daughter of Ranganayaki looked so lovely in her skirts that Komalatammal without a minute of hesitation decided that Janaki was the girl for Chinna. Ranganayaki knew that Chinnaswami was unemployed and had no future but by the union of Janaki and Ramanujan, the long friendship of Ranganayaki and Komalatammal would be everlasting. Rangaswami had reservations but he did not interfere much. On return to Kumbhakonam, ··when Komalatammal mentioned the matter to her husband, Srinivasa did not like the idea as another mouth to feed would mean more starvation. But she was unperturbed and told her neighbours that she had arranged her son's marriage and only she was responsible for it. So in July, 1909 the marriage party left for Rajendrum. The train reached Marudur very late, and by the time the party reached the bride's home, the *muhuratham* (the auspicious moment) for performing the wedding ceremony was already past. And then many inauspicious things happened like heavy gales uprooting trees, torrential rains, and damage to crops and houses. This confirmed Janaki's father in his suspicions about a happy life for her after marriage with Ramanujan, so he could not agree to *kanyādānam* (giving the daughter's hand). However, others interevened and Komalatammal had her way-the marriage was solemnised on July 14.

Janaki being too young, did not join her in-laws immediately, but the very fact of getting married added to Ramanujan's worries. He did not give up his mathematics, but realising the need for earning some money, he in a moment of dejection, told a close friend "after all it does not fill my belly". Besides, his health was never very good - he had suffered from small pox as a child, then during college days he had fallen seriously ill at least twice, and now after wedding he was frequently ill. Searching for some job in Madras, one day in 1909 he came to see his collegemate Radhakrishna Iyer in a horse carriage as he was too seriously ill to walk down. The friend put him in bed and asked the cook to look after the patient. But when his condition did not improve, Dr. Narayanaswami, who was treating him advised that Ramanujan be sent to his parents as he required con-

stant nursing. So Radhakrishna Iyer took him to the Madras beach railway station and put him on train for Kumbhakonam, requesting the Superintendent of Railway Mail Service, who was travelling by the same train to take care of Ramanujan. Before the train left, Ramanujan handed over two thick Note books to his friend and said that they were for safe keeping. "If I die, please, hand them over to Singaravelu Mudaliar of Pachaiyappa College or Edward Ross of Madras Christian College", Radhakrishna Iyer was very happy when after some months Ramanujan, returned to Madras after recovery he took the Notebooks back. Then in 1910, he had to undergo an operation for hydrocele, at that time supposed to be a major and expensive operation. But an eminent surgeon, Dr. Kuppuswami performed it without charging any fees as he was one of those few people who had recognised Ramanujun's potentiality as a great mathematician. But Ramanujan was such an active spirit that he could not remain on sick bed for long - one day he walked a distance of 9 kilometers with a childhood friend even before the stitches had healed with the result that he bled profusely.

He came back to Madras again in 1910 and stayed for a while with K.S. Viswanāth Shāstri, who had taken tuition from him in Mathematics at Kumbhakonam. Although there were many students from the same town, yet Ramanujan did not have much success in securing tuitions. Tired after whole days of tuition-hunting Ramanujan one night bemoaned "Galelio died at the hands of Inquisition and I will die in poverty". Viswanath Shastri recalled that when he was taking tuitions in Geometry and Algebra, Ramanujan would not be satisfied with giving stereotyped solutions-but would gradually lead him to the intricacies of the subject and raise the level of understanding. When Viswanath forgot any step, Ramanujan used to give another method simpler than what he had given earlier. It may not be out of place to mention here the example of another great mathematician - Harish Chandra, who was born at Kanpur (Uttar Pradesh), later settled down at Princeton (United States) and died on October 16,1983. Once during his lecture someone from the audience pointed out "Professor, you had gived a different proof last time".

Pat came the reply "Youngman, I do not give the same proof twice."

Besides giving tuitions, he was trying frantically to get a clerical job also. One day he came to see S. Balakrishna Iyer, Assistant Mathematics Master at Model High School Teachers College, Saidapet, Madras with a letter of recommendation from P.L. Seshu Iyer, and requested him to put in a word with Vice Principal Dodwell for a clerical post. Dodwell was under orders of transfer as Curator of Central Records Office near Egmore Railway Station. Dodwell said he did not know what staff he was having but promised to remember. On assuming charge of the new office he was reminded three-four times and every time said nice and sympathetic words about Ramanujan, but always gave an evasive reply. Finally Balakrishna Iyer stopped seeing Dodwell and sent a letter of regret to P.L. Seshu Iyer. Again in the hope of finding some employment, Ramanujan went to Tirukoilur, a small town about 400 kilometers from Madras in present day Andhra Pradesh to see V. Ramaswami Iyer. Ramaswami was posted there as a Deputy Collector, but because of his keen interest in Mathematics was called Professor although he never taught at any college. He was amongst the founders of Analytical Club which later on was re-christened as the Indian Mathematical Society. He also was one of the two Secretaries of the Society at its inception. Ramanujan had high hopes from him for getting at least a petty clerical job in some subordinate revenue or municipal office through his help. So he showed the Notebooks proudly. Ramaswami was amazed but felt that an "unusual genius" should not be allowed to rot at a small place, so he thought it proper to send him back to Madras with letters recommendation to some eminent mathematical workers. Once again in Madras, he had to depend upon the hospitality of his friends. It is an irony of fate or an interesting commentary on the education-employment relations that someone who had topped the list of candidates in the Final Primary School Examination of Tanjavoor District and had passed the Matriculation Examination of Madras University in the First Division was turned away from every office where there was 'no vacancy' of petty clerk available to him while his Primary School Certificate had stated more than a decade ago that "S. Ramanu-

jan is hereby declared to have qualified to admission in Public Service in accordance with the provisions of the Public Service Notification."

His parton and guide, P.V. Seshu Iyer once again advised him to try his luck at Nellore, the headquarter of a district of the same name, about 180 kilometers away from Madras, where Dewan Bahadur R. Ramachandra Rao was the Collector. In fact he was one of a small number of Indians who at that time could be given independent charge of administration of a district. Not only being an able administrator for which he had earned the title Dewan Bahadur by the British-Indian Raj, but also he was well known for his love and patronage of mathematics. As luck have it, a schoolmate, R.C.V. Rajagopalachariar offered to introduce Ramanujan to Ramchandra Rao's newphew, Krishna Rao. So one day when Ramachandra Rao was in Madras the three went to see him. He was not very much impressed by the appearance of the poor shabbily dressed half-starving lad from Kumhakonam, although the Notebooks and the letter fram P.V. Seshu Iyer were trying to tell a different story. However, out of courtesy he gave a patient hearing and kept the paper with him. After a few days when they went to see him, Dewan Bahadur said that he did not make out anything, and asking them not to bother him again, left in a hurry. Ramanujan was very much dejected and wanted to leave Madras but Rajagopalachariar persuaded him to stay on for one day more and then himself went to meet Ramachandra Rao. This time Rajagopalachariar had not taken Ramanujan with him, so, Ramachandra Rao had no hesistation in saying 'Look youngman, Ramanujan may not be a *moral fraud*, but I feel he is an *intellectual fraud*". But in the meantime Ramanujan had shown a letter to Rajagopalachariar that Professor Saldhana had written. Earlier Ramanujan had complained that he had sent some results from his Notebook to the Indian Mathematical Society, but there was no response. But now, Professor Saldhana had written on the margin of the two sides of Ramanujan's paper that the results were all wonderful, but they were not worked out step by step and were quickly arrived at from the given question itself. When this letter was shown to Ramachandra Rao, the situation changed

suddenly and he finally agreed to see Ramanujan. Later on recollecting his meeting with Ramanujan, Ramachandra Rao wrote.

"He was miserably poor. He had run away from Kumbhakonam to get leisure in Madras to pursue his studies. He never craved for distinction. He wanted leisure ; in other words, that simple food be provided for him without exertion on his part and that he should be allowed to dream on...He opened his book and began to explain some of his discoveries. I saw quite at once that there was something out of the way but my knowledge did not permit me to judge whether he talked sense or nonsense. I asked him to come again and he did and then he had gauged my ignorance and showed me some of his simpler results. Then step by step he led me to elliptic integrals and hypergeometric series and at last his theory of divergent series not yet announced to the world converted me. I asked him what he wanted. He said he wanted a *pittance to live on so that he might pursue his researches*".

From then on Ramachandra Rao tried to help Ramanujan in whatever way possible. But immediately after the meeting he sent him back to P.V. Seshu Iyer saying that it was cruel to let an *intellectual giant* like Ramanujan to rot in a town like Nellore, He also arranged to pay a monthly sum of Rs. 20-25 which Ramanujan collected for about a year from Ramachandra Rao's home in Madras. Besides, Ramachandra Rao tried to introduce his work to some influential people also, and even got for him a clerical job for 40 days at the Accountant General's Office in a temporary vacancy on Rs. 20/- per month.

This period of seven years, Deeember 1904 to December 1911 was a period of struggle for survival and frustrations for Ramanujan, and a writer K.R. Rajagopalan suggests that "It is best one does not dwell too long on these uneventful and frustrating days and pass on to the next phase when things looked somewhat more promising." But according to Ramanujan's biographer, S.R. Ranaganathan," this is the period of superactivity....His research marched on undettered by any environmental factors-physical, economic or social. Magic Squares, Continued fractions, Hypergeometric series, Properties of Numbers-Prime as well as Com-

posite, Partition of Numbers, Elliptic Intergrals and several such regions of Mathematics engaged his thought. But during that time or earlier hardly any thought was given in the country to some of these problems... Everything had tobe done and discovered by him *de novo* .. Surely all this could not be siezed by intellect alone. Intuition should have played a large part in this period of super activity. Ramanujan was inded a *drashta* (a seer) in Mathematics."

And a period of super-activity it definitely was. On the one hand P.V. Seshu Iyer was encouraging him to contribute original questions to the *Journal of Indian Mathematical Society*, and on the other his first long article on "*Some properties of Bernoulli's Numbers*" was communicated during this period. It took quite some time to appear in print. It was returned by the Editor more than once as the style of writing was felt to be "so terse" and the presentation was so lacking in clearness and precision that an ordinary reader, unaccustomed to such intellectual gymnastics would hardly follow him". Finally Ramanujan was able to give a form "suitable for publication"and the article appeared in the December Volume of the *Journal* in 1911.

On the other hand, his Notebooks bear permanent testimony to his creativity. Most of the results contained in Notebook I were discovered during his college days and this so-called period of frustration. Notebook II basically gives revised and enlarged version of Notebook I. The Notebook is full of such results, as Ranganathan has mentioned. For example Chapter XIV of Note book I contains 33 results on continued fractions and their corollaries. Entry 16 on page 107 states.

$$\frac{(x+1)^n-(x-1)^n}{(x+1)^n+(x-1)^n} = \frac{n}{x+}\ \frac{n^2-1^2}{3x+}\ \frac{n^2-2^2}{5x+}\ \frac{n^2-3^2}{7x+}\ \text{etc.}$$

$$= A_n \ , \quad A_n + \frac{1}{A_n} = \frac{2}{A_{2r}}$$

$$\text{Cor. 1 arctan } \frac{1}{x} = \frac{1}{x+}\ \frac{1^2}{3x+}\ \frac{2^2}{5x+}\ \frac{3^2}{7x+}\ \text{etc.}$$

$$2 \log \frac{x+1}{x-1} = \frac{2}{x-} \frac{1^2}{3x-} \frac{2^2}{5x-} \frac{3^2}{7x-} \text{ etc.}$$

$$3 \tan \frac{1}{x} = \frac{1}{x-} \frac{1}{3x-} \frac{1}{5x-} \frac{1}{7x-} \text{ etc.}$$

$$4 \frac{e^x-1}{e^x+1} = \frac{x}{2+} \frac{x^2}{6+} \frac{x^2}{10+} \frac{x^2}{14+} \text{ etc.}$$

He does not give any proof of the statements or the corollaries, but this is what Ramanujan's style was, and as Professor Hardy puts it "it is perhaps in his work in these fields that Ramanujan shows at his very best".

5

Spring Brings New Life

Can a self-respecting hard-working person live on others hospitability ? No. Not for long. Ramanujan did accept help of Rs. 20/-per month for a year, but he was always trying to secure some clerical job to earn even a few rupees which he could feel was his honest earning. On the other hand Dewan Bahadur Ramachandra Rao was also trying to use his personal influence to find some work at some office for Ramanujan. This is how Ramanujan got a temporary clerk's job in the Accountant General's office. Then he came to know of a vacancy in the office of Chief Accountant, Madras Port Trust. So, on February 9, 1912, in his application he wrote "I have passed the Matriculation Examination and studied upto F.A. but was prevented from pursuing my studies further due to certain untoward circumstances. I have, however, been devoting *all my time to Mathematics and developing the subject.* I can say I am quite confident, I can do justice to my work if I am appointed." He knew full well that his *expertise* in research was of no relevance to the petty mechanical job for which he was applying, but Mathematics was so much dominant in his thoughts that he could not refrain from emphasising it.

Three weeks later, he got the job and on march 1, 1912, Ramanujan joined Madras Port Trust in the Accounts Section in Class III Grade IV on Rs 30/- per month. This *post* meant much. It was 50% rise in monthly pay, the job was not temporary. So

Ramanujan called his mother to join him in Madras and soon his wife Janakiammal also joined. At work-place he came in contact with the Manager of the Port Trust S. Narayana Iyer, who was also the Assistant Secretary and Treasurer of Indian Mathematical Society for a number of years since its inception. He not only did provide Ramanujan all facilities to work on his mathematical problems, but also brought his talents to the notice of the Chairman Spring. Sir Francis Spring was an engineer by training and belonged to that class of British administrators who were really interested in the welfare of the Indians whom they were expected to serve unlike many of his counter-parts connected with the so-called steel-frame of British Raj - the Indian Civil Service–who thought their main job was to stifle the rising wave of national movement. It was this devoted administrator who designed and developed the Madras Harbour and built a bridge across the river Godavari. He would provide all possible help to any Indian if he found some talent in him. So he decided that Ramanujan would work under the direct supervision of Narayana Iyer. Narayana Iyer would not only not abject to Ramanujan doing Mathamatics during office hours, but also discussed mathematics with him in the evenings. It is said that one day when Ramanujan sent the files to Spring, two sheets containing some mathematics, that Ramanujan had done that day also were inadvertently included. Francis Spring was surprised at the 'hard Mathematics'. He knew that Ramanujan was a mathematician who had already made some contribution to the journal of Indian Mathematical Society but he did not realise earlier what quality of Mathematics his poor clerk was doing. So Ramanujan was called the next day by Spring. Earlier when Ramanujan realised what had happened he was very much worried and returning home told his mother about it. She immediately performed *Pujas* (worship) and invoked the Deity to pretect her son. When a worried Ramanujan appeared before him, Spring feigned anger and 'rebuked' him for doing 'private work' during office hours but after a moment asked him to sit at another place where he could go on with his mathematical works undistributed. He was from now on to submit his weekly notes and jottings through the Manager. A miracle had happened, so thought the mother while Ramanujan expected sack, his talent

was recognised by his boss. She made further offerings as a mark of Thanksgiving. So Ramanujan became busier at the Port Trust, but his financial condition did not improve. He could not even buy paper for noting down his results. Often he would take plane, used or unused packing paper and jot down his results. He continued to give tuitions to buy a few bags of rice, and would come home very late at night. Poor Janakiammal would be waiting for her husband and when he would be very late, she would cry and the mother-in-law would admonish her saying "How can you fill your belly if Chinna does not work so hard".

The other benefactor of Ramanujan, Dewan Bahadur also was trying to get Ramanujan's talent recognised by the people in Madras whose voice counted. So he wrote to C. L. T. Griffith, Mathematics Professor at the College of Engineering Griffith saw some results of Ramanujan and was very much impressed. Before making any definite assessment of the results that he saw, he decided to write a letter of recommendation to Sir Francis on 12th November, 1912 saying "you have in your office as an accountant a youngman named S. Ramanujan who is a most remarkable mathematician. He may be a very poor accountant, but I hope you will see that he is kept happily employed until *something can be done to make use of his extra ordinary gifts.* I am writing to one of the leading mathematicians at home about him and sending copies of some of the Ramanujan's papers and results...If there is any real genius in him, he will have to be provided with money for books and with leisure but until I hear from home, I don't feel sure that it is worthwhile spending much time or money on him."

The same day Griffith wrote to Professor M.J.M. Hill who had taught him Mathematics twenty years ago at the University College of London University. Hill's response was just what could be expected of an elderly professor. He promptly replied promising to look into Ramanujan's paper on Bernoulli's numbers which had already appeared in the *Journal of Indian Mathematical Society* when he had time after the end of the term but felt that Ramanujan had fallen into the pitfalls of divergent series. Otherwise he could not get the erroneous results you send me

$$1+2+3+\text{etc} = -\frac{1}{12},\ 1^2+2^2+3^2+\text{etc} = 0,$$

$$1^3+2^3+3^3+\text{etc} = \frac{1}{240}.$$

All these three series have infinity for their sum. The book which will be most useful for him is Bromwich's *Theory of Infinite series*" Then he gave some suggestions about the style of writing so that Ramanujan's papers may be considered for publication... "He should not use symbols which he does not explain...What you say about him personally is very interesting and I hope something may come out of his work "

Professor Hill (1856-1926), who was born in Berhampur, a town in India was not unsympathetic to Ramanujan—he even agreed that the results 'intuitively obtained' were interesting. But like most practising mathematicians he was hesitant to accept anything out of the conventional beliefs. He was trying to apply principles of convergent series on divergent series, that is possibly the reason why he suggested a study of Bromwich's book. Such situations do occur when a new theory violating all "norms of common sense" is presented. Only two decades earlier when George Cantor in his *Mengenlehre* had demolished the two thousand year old belief that "the part is smaller than the whole" by showing among other arithmetical and geometric situations that "a very small line-segment has the same number of points as a line of unlimited length" the conscience of most mathematicians was outraged. Kronecker called it "a dangerous type of mathematical insanity" And in 1908, at the International Congress of Mathematicians, Rome, Pioncare prophesied that "Later generations will regard *Mengelehre* as a disease from which one has recovered". And we all know that the later generations have not "recovered" from the *disease* "Set Theory" (Mengelehere) but we are getting more and more afflilicted by it.

Griffith informed Spring about Hill's comments on Ramanujan's work and in a letter suggested that R. Ramachandra Rao may be asked to buy Bromwich's book for Ramanujan who may be advised to give proper and detailed proofs of his results. It

is not known whether the book was purchased and made available to him or not but Ramanujan was no longer dependent on Carr's *Synopsis*. He used to borrow higher books wherever he could. Prof. K B. Madhava, who used to correct galley proofs of the *Journal Indian Mathematical Society* had a library of his own, and Ramanujan used to borrow books from him. Madhava would lend reluctantly as he thought the 'higher' books beyond Ramanujan's capacity even his paper on Berneulli Numbers did not make any difference because Madhava knew that it was corrected by many people.

In the meantime Spring was trying to get Ramanujan recognised by the local community of mathematicians and for this purpose he sought help from A.G. Bourne, Director of Public Instruction. Bourne's response was that of a typical high government officer. In his letter of 14th November, 1912, he wrote to Sir Francis "I never heard of Ramanujan, Littlehailes is now away on the west coast, Middlemast is at Presidencey College and Graham at the Fort. I should certainly send him to see one or both of these. I do'nt like the suggestion that access to a library would ruin any genius; it savours of the middle ages and if his genius is so elusive or mysterious that good mathematicians possessed besides of much common sense cannot recognise and appreciate it even if it carries them beyond their scope, *I should doubt its existence.*" However, Ramanujan saw W. Graham, at the Accountant General's office in Fort St. George on November 27, 1912 who wrote to Griffith' "He seems to have done a great deal of work in one particular branch of calculus and from the way he has done it, he must have considerable mathematical aptitude. Whether he has the stuff of great mathematicians or not, I do not know. He gave me the impression of having brains. His original work is interesting development of work already done-but interesting only to the purist. However I am not the best qualified. Middlemast's opinions would be of value".

But Spring could not be satisfied with this 'qualified recognition' by the small circle of mathematicians based in Madras. He wanted Ramanujan's potentiality to flourish and be recognised by the wider community of mathematicians. So with his advice

and Seshu Iyer' help, Ramanujan started correspondence with Cambridge mathematicians. He communicated some of his results to a distinguished analyst and geometer, Professor F.H. Baker and also to Professor E.W. Hobson. The two very senior mathematicians did not attach any importance to the so-called Theorems' which Ramanujan had very proudly presented. They simply returned the manuscript without any comment. There is nothing surprising in this. Even today a paper is returned by a Journal after being kept for a few months, only to find a place in some more reputed journal.

And then began correspondence, exchange of ideas, co-operation and collaboration with Hardy. Hardy, who was only ten years senior to Ramanujan in age, had due to favourable circumstances already become a reputed mathematician while poor Ramanujan, who was a petty clerk was struggling to 'do' some mathematicis In his first letter dated January 16, 1913, Ramanujan after introducing himself as a clerk, 23 years old with no University education wrote "I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by local mathematicians as *startling*...Just as in elementary mathematics you give a meaning to n when n is negative and fractional to conform to this law which holds when n is a positive integer, similarly the whole of my investigations proceed on giving a meaning to Eulerian Second Integral for all values of n. My friends who have gone through a regular course of university education tell me that $\int_0^\infty x^{n-1} e^{-x} dx = \Gamma(n)$ is true only when n is positive. They say that this integral relation is not true when n is negative. Supposing this true only for positive values of n and also supposing the definition of $n\Gamma(n) = \Gamma(n+1)$ to be universally true, I have given meaning to these integrals and under the conditions, I state the integral is true for all values of n negative, and fractional. My whole investigations are based upon this and I have been developing it to a remarkable extent so much so that the local mathematicians are not able to understand me in my higher flights".

In the same letter he said "Very recently, I came across a tract published by you entitled *Orders of Infinity* on page 36 of which I find a statement that no definite expression has been as yet found for the number of prime numbers less than a given number. I have found an expression which approximates to the real results, the error being negligible. I would request you to go through the enclosed paper...I have not given the actual investigations nor the expressions that I get, but I have indicated the lines on which I proceed. Being inexperienced I would very highly value any advice you give me." With this covering letter he enclosed statements of about a hundred results dealing with Number Theory, Definite Integrals, Summation of Series, Transformation of Series and Integrals, Approximate Integration and summation of series and Continued Fractions.

The initial response of Geofrey Harold Hardy, of Trinity College, Cambridge, a world famous mathematician, could not be any different from that of his other colleagues - Baker and Hobson. As C.P. Snow describes the event in his Introduction to Hardy's *A Mathematicians Apology* "One morning early in 1913, he found, among the letters on his breakfast table, a large untidy envelope decorated with Indian stamps. When he opened it, he found sheets of paper by no means fresh, in which, in a non-english holograph, were line after line of symbols. Hardy glanced them without enthusiasm...He glanced at the letter in halting English, signed by an unknown Indian, asking him to give an opinion on these mathematical discoveries. The script appeared to consist of theorems, most of them wild or fantastic looking, one or two already well-known, laid out as though they were original. There were no proofs of any kind. Hardy was not only bored, iritated. It seemed like *a curious type of fraud*. He put the manuscript aside and went on with his day's routine". But, as Bruce C. Berndt has noted, that evening Hardy and his colleague and scientific collaborator J. E. Littlewood "retired to the chess room over the common room of Trinity College. Before they entered the room Hardy exclaimed that Hindu correspondent was either a *crank* or a *genius*. After two and a half hours they emerged from the chess room with the verdict *genius*".

So it was the second time that Ramanujan was suspected to be a fraud and then recognised as genius - first by his campatriot, Dewan Bahadur R. Ramachandra Rao and now by a Cambridge personality, Professor G.H. Hardy. And both of them became ardent admirers and supporters of Ramanujan.

Hardy sat down to work; he alongwith Littlewood began a thorough examination of Ramanujan's results given in the "enclosed papers" and tried to work out a plan for having Ramanujan at Cambridge. Accordingly at his instance the Tutors of Trinity College conveyed their desire to the Secretary for Indian students in London supporting Hardy's plan. So on February 3, the Secretary sent a letter to Arthur Davis who was Secretary to the Advisory Committee for Indian Students in Madras. On March 10, Davis informed B. Hanumantha Rao, Professor of Mathematics, Madras Engineering College, about the Tutors' desire to have Ramanujan at Trinity College. And then Hardy wrote to Ramanujan pointing out that some results were very interesting but many could not be verified as no proofs were indicated on how the results were arrived at. As was expected Hardy's letter opened a new chapter in Ramanujan's life. It was very proudly shown to Spring, who must have been happy that his efforts towards Ramanujan's genius being recognised were moving towards success. As luck have it, Sir Gilbert Walker happened to be in Madras. Sir Gilbert was not only a top officer of British-Indian Government (he was the Director General of Observatories with headquarters at the Summer Capital of the Government, Shimla), but also a Fellow of Royal Society and formerly a Fellow of Trinity College, Cambridge. Sir Francis Spring showed him the Notebooks of Ramanujan and Hardy's letter. He was amazed at the work that Ramanujan had put in braving all sorts of impediments. So on 26th February, 1913 he wrote to Francis Dewsbury, Registrar, Madras University "the character of the work that I saw impressed me as comparable in *originality with that of a mathematics fellow in a Cambridge College;* it appears to lack, however, as might be expected in the circumstances the completeness and precision necessary before the universal validity of the results, could be accepted,...It was perfectly clear to me that the University would be justified in enabling S.Ramanujan for a few

years at least spend the whole of his time on mathematics without any anxiety as to his livelihood. I would suggest that they should communicate with Mr. G H. Hardy, Fellow of Trinity College, Cambridge, with whom he is already in correspondence and assure Mr. Hardy of their interest in him."

With this strong letter of recommendation by Sir Gilbert, all the hurdles for the University, whose only job was to *examine the students* enrolled at colleges, in doing something for a person to carry on his *research* who had failed in F A . melted in the air. Immediately the University machinery started with great speed. On March 13, 1913, Professor B. Hanumantha Rao of Engineering College, wrote letters to the members convening the Board of Studies' meeting. He invited S. Narayana Iyer also to attend the meeting in order to consider "what we can do for S. Ramanujan. Come and assist us as you have seen his results." The Board of Studies in its meeting recommended award of scholarship of Rs.75/- per month to S. Ramanujan for two years. And then the matter went to the Syndicate of the University. The Vice Chancellor placed the letter from Sir Gilbert Walker and the recommendation of the Board of Studies. But many members of the Syndicate opposed the move citing the University Regulations on Research Scholarship, which prescribed possession of a Naster's degree as a necessary condition. For some time, as Ranganathan says, itappear ed as if the Vice Chancellor's proposal would be rejected. But a member, Mr. Justice P.R. Sundaram pointed out that"as stated in the Preamble to the University Act, one of the Johs of the University was to promote research. Here was a candidate of proven phenomenal capacity for research. The University could be failing in its duty if it did not use him as a means for the promotion of research. The Regulations were University-made. *Any regulation obstructing the implementation of the provisions of the Act of the Legislature should be deemed to be ultra vires.*" Finally the Syndicate "resolved that the recommendation of the Board be accepted and that sanction of the Government to payment of the proposed scholarship from the University Fee Fund be requested in accordance with the provisions of Section XV of the Act of Incorporation and Section 3 of Indian Universities Act, 1904."

On April 7, Government approved the Syndicate's proposal. Two days later the Registrar informed Ramanujan about award of the scholarship. Accordingly he was required to spend full time on mathematical research and submit a report of his findings every three months. He would be entitled to use of University Library. Without hesitation Ramanujan sent his acceptance to the Registrar. Sir Francis granted him two years' leave without pay from the Port Trust. May 1, 1913 became a *red letter day* in the annals of Madras University That day Ramanujan joined as a research scholar who was destined to become a world-phenomenon and then secondly the character of the University also changed *from a purely examining body it became a centre of research also.*

After a few days of joining Madras University, Ramanujan, sent his mother and wife home to Kumbhakonam and went to live in a small house. Now he used much of his time in one of the alcoves of the University library. In the evening he would sit with a slate and pencil at Triplicane beach and afterwards he would go to S. Narayana Iyer's house and every night both worked till late in the night, first writing on slates. Sometimes Ramanujan would get up at 2.30 in the morning and jot down solutions that he claimed to have found in his dream. Ramanujan's slate is still with the family of Narayana Iyer who got it when Ramanujan was leaving for Cambridge.

Ramanujan himself had no idea that at the recommendation of Gilbert Walker, as master-minded by Francis Spring, the University of Madras would go out of its way to help him. He, in fact, had pinned all his hopes on Hardy. So in his second letter to the Cambridge Don, dated 27th February, he wrote "I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed. To preserve my brains, I want food and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from the Government...I find in many a place in your letter rigorous proofs are required. I told him (?) that the sum of an infinite number of

terms of the series $1+2+3+\ldots = -\frac{1}{12}$ under *my* theory. If I tell you this you will at once point out to me the lunatic asylum is the goal. What I tell you is this. Verify the results I give and if they agree with your results you should at least grant that there may be at least some truth in my fundamental basis." Alongwith this letter, Ramanujan sent the statement of some more Theorems and this time added some results on Hypergeometric Series also.

It may be mentioned here that Ramanujan's Theory of Divergent Series has been spelt out in Chapter VIII of the Notebook, Vol. I pages 40-41 and again given in Chapter VI of Vol. II pages 59-60. It is as follows :

"$f(1) + f(2) + f(3) + \ldots\ldots + f(x) = \phi(x)$

$$\phi(x) = C + \int f(x)\,dx + \frac{1}{2} f(x) + \frac{B_2}{2;} f'(x) - \frac{B_4}{4!} f''(x + \ldots$$

$$+ \frac{B_6}{6!} f^{v}(x) - \frac{B_8}{8!} f^{vii}(x) \quad \ldots\text{etc}$$

Here Ramanujan's B_{2n} is Bernoull is number B_n in standard notation).

"In giving any values to x, C can be found, R H S. is not a terminating series except in some special cases. Consequently no constant can be found in $\frac{1}{2} f(x) + \frac{B_2}{2!} f'(x) - \frac{B_4}{4!} f'''(x) +$ etc. except in those special cases. If R.H.S. be a terminating series it must be some integral function of x; in this case there is no possibility of a constant (according to the ordinary sense) in $\phi(x)$; for $\phi(1) = f(1) + \phi(0)$. But $\phi(1) = f(1)$. Therefore $\phi(0)$ is always 0 whether $\phi(x)$ is rational or irrational. Thesefore when $\phi(x)$ is a rational integral function of (x), it must be divisible by x since $\phi(0) = 0$ and consequently no constant but 0 can exist. Therefore let us try *to give some meaning for the constant of a series.* The constant of a series is the constant obtained by completing the remaining part by faithfully adhering to the above rule for summation of series. The constant of a series has some mysterious connnection with the given infinite series and it is like

the centre of gravity of a body. Mysterious because we may substitute it for the divergent infinite series.

"Now the constant of the series $1+1+1+1+1+$ etc is $-\frac{1}{2}$; for the sum to x term $= x = C + \int 1\,dx + \frac{1}{2}$. Therefore $C = -\frac{1}{2}$.

Another way of finding the constant is as follows – let us take the series $1+2+3+4+5$ and c. Let C be its constant. Then

$$C = 1+2+3+4\ldots \text{ etc}$$
$$\therefore\ 4C = \quad 4+8\ldots \text{ etc}$$
$$\therefore -3C = 1-2+3-4\ldots \text{ etc} = \frac{1}{(1+1)^2} = \frac{1}{4}.$$
$$\therefore\ C = -\frac{1}{12}$$

"For finding the sum to a fractional number of terms assume the sum to be true always and is anything difficult in finding $\phi(h)$ where h is small, take n any integer you choose, find $(n+h)$ and then subtract

$f(1+h)+f(2+h)+\ldots+f(n+h)$ from the result. "The sum to a negative number of terms is the sum with the sign changed, calculated backwards from the term previous to the first to the given number of terms with positive sign instead of negative."

These methods are illustrated in Notebook v. II on page 61 : for example $1+2+3+\ldots$etc to -5 terms $= -\ (0-1-2-3-4)$ $= 10$ and $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{h} = C_o + \log_e n - (\frac{1}{1+h}+\frac{1}{2+h}+\ldots$ $+\frac{1}{n+h})$ where n is any [illegible]eger and C_o is the constant of $\Sigma\frac{1}{n}$"

It may be seen that the definitions and illustrations both suffer from lack of clarity and precision which make it difficult to understand Ramanujan's theory. However a few years later Hardy utilised the formula in the case of infinite number of terms (not upto any negative or fractional number) in his *Divergent Series* (pp. 327, 346-346) with some modification.

However, on receipt of Ramanujan's letter of 27th February, in which he had given some idea about the method of proof, Hardy sent the papers to Littlewood. On hearing from Littlewood, Hardy wrote to Ramanujan on March 26,1913 mentioning amongst other things "Mr. Littlewood suggested to me that your unwillingness to give proofs was probably due to apprehensions as to the use I might make of your results. Let me put the matter quite plainly to you. You have in your possession now 3 long letters of mine, in which I speak quite plainly about what you have proved or claim to be able to prove. I have shown your letters to Mr. Littlewood, Dr. Barnes, Mr Berry and other mathematicians. Surely it is obvious that, if I were to attempt to make any illegitimate use of your results, nothing would be easier for you than to expose me. You will, I am sure, excuse my stating the case with such bluntness. I should not do so if I were not genuinely anxious to see what can be done to give you a better chance of making the best use of your obvious mathematical gifts."

There was no arrogance or *patronisation* in this letter. It simply indicated a real scholar's desire to help another scholar and thus help in the advancement of Mathematics. Obviously Ramanujan's reaction could be as expected. No inferiority complex, no entreaties, no *mai-bap* as was the attitude of many Indians towards Britons those days. So in reply, he opened his heart as a self-reliant Indian when in his letter of April 17,1913 he wrote "I am a little pained to see what you have written...I am not in the least apprehensive of my method being utilised by others. On the contrary my method has been in my possession for last eight years and I have not found anyone to appreciate the method. As I wrote in my last letter I have found a sympathetic friend in you and am willing to place unreservedly

in your hands what little I have. It was on account of the novelty of the method I have used that I am a little diffident even now to communicate my own way of arriving at the expressions I have already given."

In the same letter, he informed Hardy about award of Research Scholarship from Madras University and also expressed his thanks to the Cambridge mathematicians who had shown interest in his work. More letters were exchanged and Hardy was trying more and more to get him in Cambridge, so that this *pen-friendship* may develop into scientific collaboration. But Ramanujan was not prepared to travel to Britain Who knows what the reasons were : allergy of some orthodox Hindus to cross the oceans, or family constraints, or his eating habits and unstable health or any other factor. But ostensibly the cause would have been that he had already got from Madras University what he wanted – ' a pittance to survive and opportunity to dream on'. Travel to Britain would mean distraction and hindrance to his research. After all he was working without any 'worries' and was getting sympathetic suggestions from Hardy.

As required by the University, Ramanujan had to submit quarterly reports of his findings. It is known that he submitted three Reports dated 5th August, 1913, 7th November 1913 and 9th March 1914. One would expect the Reports to be kept in the Archives of Madras University. However copies are on file in the library of Trinity College. These Reports have not been published, but Hardy has mentioned some results contained therein. Recently Bruce C. Brendt has analysed them in a paper which has appeared in the *Bulletin of London Mathmatical Society* 16(1984) 449-489. The first two reports and a portion of the third are concerned with certain integral formulas connected with the theory of integral transforms. The result of this category has been called Ramanujan's Master Theorem. The Theorem states :

Suppose that in some neighbourhood of $x=0$,

$$F(x)=\sum_{k=0}^{\infty}\frac{\phi(k)\,(-x)^{k}}{k},$$

Then $\int_0^\infty x^{n-1} F(x)\, dx = \Gamma(n)\phi(-n)$.

In proving this Theorem Ramanujan assumes that F(x) can be expanded in Maclaurin Series, and is continuous on $(0, \infty)$, also n is positive and $\lim_{x\to\infty} x^n F(x) = 0$, but the last condition can be relaxed if F is a (—) circular or Bessel function. He cites a number of examples, some given in a paper entitled "Some Definite Integral" which appeared in the *Messenger of Mathematics* 44(1915). An example is given in the Note book vol. II p. 42. which has many applications in Transformation. Theory :

Let m, n be positive and $x = y/(1+y)$ then

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_0^\infty y^{m-1} (1+y)^{-m-n} dy.$$

In the final section of the first Report Ramanujan establishes a sort of Converse to the Master Theorem. Berndt points out the results can be obtained by using Inversion formula for Mellin's Transform, although Rajmanujan used purely formal methods.

In the second Report Ramanujan gives two more applications of his Master Theorem and derives several related expansions using the Converse Theorem. In the Third report, besides giving a few more applications of his Master Theorem, he introduces the "fractional order" of functions. We know that by composition $F^m(F^n(x)) = F^{m+n}$ where both m and n are non-negative integers. Ramanujan asserts that this is true even when m and n are real numbers. This led to various ramifications like fractional differentiation, fractional integration and difference operators. It appears that Ramanujan was in a hurry *to produce* results for the University-all results were obtained using formal methods, some correct and some already established much earlier by others with mathematical rigour. But he did not forget Hardy's tract *Order of Infinity* which actually set the game of Hardy–Ramanujan

collaboration rolling. He found some results dealing with orders of infinity and introduced what he called *scales of infinity*.

And what was Hardy doing apart from writing long letters to Ramanujan ? He did not believe in *soft analysis* as he was an advocate of *hard analysis*. He was trying his technique to get a reluctant Ramanujan to Cambridge and finally was able to "solve" the problem which was bedeviled with numerous constraints.

6

Amman : Seven Seas

Someone has said

East is East and West is West
And never the twain shall meet.

But where is the line on the Globe that *seperates* the twain. Australia claims to be West while Malaysia which is really to its west is termed East and America which is situated further eastwards is called West. On the other hand Indians are told by their elders *Vasudheya Kutumbakam* (the Universe is a family) and

Banī Adam a'zā-i-yak-dīgar and
ki dar afrīnish ze yak jauhar and

(Adam's children are limbs of one another, because in creation they are from the same essence). But differences and distinctions do exist. They can and do exist even in the same country, the same town, the same village—tiny or expansive. It is all in the mind, a product of the environment, scientifically speaking a function of several complex variables-religion, colour, language, dress, eating habits, means of livelihood—and each component has its own real and imaginary parts.

Ramanujan who was supposed to have belonged to the East and, Spring. Littlehailes, Walker, Middlemast, Hardy and a host of others who claimed to be from the West had already met,

and Cambridge was calling Ramanujan, but he was not prepared to go. Was he reluctant or simply hesitant ? Was he afraid of the adverse reaction of the caste leaders who inspite of being the great Ramanuja's followers had become practitioners of a sort of untouchability ? Or he suspected or was made to suspect that his mathematical treasure would be plagiarised and stolen by the foreigners ? Or was he dissuaded by some elderly people because if he went, their own offspring would also emulate him and say good-gye to them? Or was it pure mistrust due to political conditions prevailing: whatever the reason, his well-wishers were making concerted efforts to send him abroad. They all took advantage of a Fellow of Trinity College, E. H. Neville's visit to Madras. Spring asked him to convince Ramanujan and Hardy wrote to Ramanujan "Try to make the acquaintance of Mr. E. H. Neville, who is now in Madras lecturing. He comes from my college and you might find his advice as to reading and study invaluable." Neille describes the event of his meeting with Ramanujan :

"After my first lecture, Ramanujan was introduced. We sat down and he turned the pages of a Note Book. Two days later he turned the pages again, and after our third meeting he said, 'Perhaps you would like to take it away with you.' The astounding compliment took away my breath. The priceless volume had never before been out of his hands; no Indian could understand it, no Englishman was to be trusted with it. The truth was, of course, that the *English were objects of suspicion not as Individuals but as components of the governing machine. I came from outside the machine* and for no reason other than that, I enjoyed an overwhelming advantage which was grossly unfair. Richard Littlehailes, the professor of Mathematics, in particular, had done and was to do far more for Ramanujan in Madras than I did . . .Ramanujan's trust being won so completely, I raised immediately the question of Cambridge, and found to my delight and surprise that Ramanujan needed no converting. . . Lest he should be harassed presently by attempts to dissuade him, I addressed myself to the task of convincing his friends that the proposal was in Ramanujan's own interest. I at once wrote to Hardy that the intangible obstacles had disappeared and. . . I do not know what account I gave of Ramanujan and the note-book, but I made it abundantly clear

rically hidden under a bushel in Madras... Mr. Neville tells me that he has already forwarded to you a memo dealing with Ramanujan; so it is unnecessary to state more than I have done... I therefore recommend that this University grant Ramanujan a scholarship of the value of £ 250 per annum for one year, to be possibly extended in whole or in part for a second year on receipt of a report from the authorities of Trinity College, Cambridge, and in addition give him a sum of £ 100 which will be necessary for his passage to England and initial outfit.

"Mr. Neville, who leaves India on March 14th, would be willing to take him to Cambridge provided a passage can be obtained in the same boat as that in which he is travelling; but if that is not possible then he would like Ramaunjan to leave at the earliest possible date so that he may reach England before the end of May term.

"I trust that the Syndicate will be able to grant this scholarship which promises to bring *credit on this University as being one of the foremost in the promotion of research and advancement of knowledge*".

The syndicate approved Littlehailes' proposal without delay and sent it for the Government's sanction. But Sir Francis was not prepared to take any chances, and leave the file lying idle in the Governor's Secretariat. So he immediately despatched a letter to Mr. C.B. Cotterell, Secretary to Lord Pentland, Governor of Madras on 5th February :

"If I understand right his Excellency has the Educational Portfolio, So I am anxious to interest him in a matter which under the circumstances is, I believe very urgent. It relates to affairs of a *clerk in my office named S. Ramanujan who, I think His Excellency has already heard from me*, is pronounced by very high mathematical authorities to be a mathematician of new and high, if not transcendental order of genius...Last evening I heard from Mr. Littlehailes and others that the University Syndicate had decided subject to Government sanction, to set aside a sum of Rs.10,000/– in order to secure Ramanujan's visit to England for a couple of years. Messers Littlehailes and Neville *begged* me to intercede

with His Excellency with a view to the speedy confirmation of this action of the University Syndicate. But I wish to make it quite clear that I write under no mandate from the Syndicate but merely as a private individual *interested in my employee Ramanujan as well as Mathematics*".

One week later, the Government sanctioned a sum of Rs. 10,000/-from the University Vacation Lecturer's fund for grant of £250/-to Ramanujan in Cambridge for two years funds for free passage and reasonable outfit. Immediately on receipt of Government sanction, the Registrar intimated Ramanujan on 13th February who sent his letter of acceptance on 16/2/1914 and requested the Registrar to arrange for monthly payment of Rs.60/-to Komalatammal at Kumbhakonam and quarterly payment of scholarship in Cambridge through Messers Barclay and Co. Bankers.

Now, when all preparations for his departure were complete, his mother and wife came to see him. Janakiammal one day expressed her desire to accompany him. She had no hesitation or fear of religious, social or caste repercussions, she simply said "I will give you comfort if I am with you". But Ramanujan knew that his mother would never agree, so to parry he replied "You are young and beautiful, I will all the time worry about you when I am at college. Some Englishman will take a fancy for you and run away with you." The poor fourteen year old bonny lassie could not say a word further. Now there was another preparation to be made-to make Ramanujan appear a Brown Englishman. He had to get his hair cut in the European style. get rid of his *Kudumi* (the tuft of hair at the back of the head) which was the symbol of his high caste, put on shoes, wear trousers, and discard cotton *dhoti* (loin cloth). But how could mother agree to all this. She had given him permission to go only at the command of Namagirithayar. While his well-wisher. Sir Francis not keen on formal aspects, had even told the Governor that he would not like to disturb Ramanujan's orthodoxy so that he could find his proper place in the community of mathematicians and bring credit to India. And there were some Indians-both young and old who thought that formal appearance was unimportant, there was no harm in changing the dress as long as one remained faithful to

one's origin and did good work. So it was decided that all this would be done when the mother was not nearby. So it was decidid to sent her to Kumbhakonam. They all went to see her off at Madras Beach Railway Station. Ramanujan performed *Shāstāng namaskār* (prostration) before his mother. Komalatammal gave *asīrwātham* (blessings), Janaki tried to hide her tears and her husband looked towards her for a moment and lowered his eyes down.

Then he did get a hair-dressing of the European style, but he felt very sad at the *Kudumi* being shaved off after all he had nurtured it for two decades. Then he did his shopping at Wren Bennet Store buying three-piece suits, neckties, shoes. socks and the like. And as R. Gopala Iyer, retired Professor of Zoology, Presidency College, Madras recalls "On the very eve of his sailing Ramanujan came to the College Assistant Professors' Common Room with a huge suitcase packed with articles of western dress. He laid them one by one on the table and sought the help of those who wore them to show how to wear them properly. He had particular difficulty in making the knot of the neck-tie". Anyway he learnt all that, put on socks and shoes and stood before the mirror. Someone took a photograph which he later sent to his mother. Poor Komalatammal had great difficulty in realising that it was her own Chinnaswami. Then he went to see Narayana Iyer who was astounded to see a Brown Englishman. But when he recognised the man for whom he had worked so hard, he hugged Ramanujan in happiness. There was one more benefactor R. Ramachandra Rao, whom Ramanujan went to see. Later, the Dewan Bahadur wrote "The last two or three days of his stay In India before he proceeded to England he spent with me in a mofussil station. I took him to the house of a friend living in European style to initiate him in the mysteries of a fork and knife under strict stipulation that nothing but vegetable food should be served. He was not happy. He did not relish food being served by strange servants. He was not very jubilant over his future and over his assured distinctions to come. He seemed to have moved as if *he was obeying a call*"

Seventeenth March, was the big day when Ramanujan was to set sail on S.S. Nevasa. Messers Binny and Co. presented to

him a Second Class ticket for the voyage, at the instance of Sir Francis who instructed the agents of the shipping company to serve only vegetarian food to Ramanujan on the boat. All other arrangements were being looked after by Littlehailes and Arther Davis. Even a meeting was held to bid farewell to an *F.A-tailed petty clerk who was now on the way to recognition by Great Britain.* It was attended by V.I.P.'s like Justice Sankaran Nair, Seshagiri, Professor Middlemast, C. Nanjunda Rao, Kasturi Ranga (proprietor of the *Hindu*), besides Narayana Iyer and Francis Spring himself. A prominent Advocate of Madras High Court, S. Srinivasa Iyengar presided.

When he was about to board the ship, there were no relatives to say *Shubh yatra* (Bon Voyage). Only 2-3 Indian friends and many Britons had some to see him off. Poor Ramanujan burst into tears An elderly member of the Salvation Army embraced him, wiped his tears and said "It is a rare honour that you are going to London. This is only by God's grace. Go, sonny, having faith in Him. My God wants success and happy return". These words of sympathy reminded Ramanujan of Sri Namagirithayar. He folded his hands, bowed his head and said his silent prayer to the Goddess. Then the Captain of Nevasa himself came to welcome him and the elderly gentleman asked him to take care of Ramanujan. He promised to do just that and said "He can come to my cabin whenever he likes and I will also visit him, but he should not pester me with mathematics".

The boat took twelve days to reach Suez, from where Ramanujan wrote to his friend. P. Krishna Rao, Dewan Bahadur Ramachandra Rao's nephew, telling him that the first three days of the voyage were uncomfortable due to sea-sickness. Later on the sea was smooth and he reached Britain safely after four weeks on the sea. The Captain also kept up his promise - he took particular care of Ramanujan and saw to it that only vegetarian food, milk and fruits were served to him. And at last when on 14th April, 1974, S.S. Nevasa berthed on a Southampton dock, a lonely Ramanujan alighted from the boat he did not know if there was anyone to welcome him. But Neville and his brother were waiting at the Harbour. They took him to their house, gave him a hearty reception, entertained him for three days and then Neville took him to Cambridge - the destination where Hardy was.

7

I Came, I Joined, I Conquered

Suppose someone from Britain, who is accustomed to take *mince-and-mealie* and *stovies* or *fish-and chips* and *steak*, comes to India with a view to stay for some length of time. Suppose further that he can get only *dal-bhat* or *chaura-moru*, what unenviable plight the poor person would be in. And then the climate, the dress and language will be the other worries. But if the visitor has come with some purpose, he will try to adjust. Ramanujan's case was isomorphic. He had never earlier put on shoes and socks, he had *failed* in English, he had not touched meat or eggs, he hailed from Tamil Nadu where the weather was steady-from warm to hot. For him even Delhi could be cold in December January. Now he was in Cambridge, where even the University was different from the University of Madras that he had known. Neville, who was greatly instrumental in Ramanujan's coming to Cambridge recalls "It was in April 1914 that Ramanujan arrived in Cambridge, where he lived in my house until residential accommodation became available for him in July in Trinity. He felt the petty miseries of life in a strange civilization, the vegetables that were unpalatable because they were unfamiliar, the shoes that tormented the feet that had been unconfined for 26 years, but he was a happy man, revelling in the mathematical society in which he was entering and idolised by Indian students". An Indian student, P.C. Mahalnobis, who later founded the Indian Statistical Institute at Calcutta and was an ardent supporter of the policy of planning and theimportant role of Planning Commission in the Republic ofIndia,

had been in Cambridge since 1913 as a student of Mathematics at King's College, noted "One day soon after his arrival, I went to see Ramanujan in his room in Trinity College...Ramanujan was sitting very near the fire. I asked him whether he was quite warm at night. He said that he was feeling the cold though he was sleeping with his overcoat on and also wrapping himself up in a shawl. I went to his bedroom to see if he had enough blankets. I found that his bed had a number of blankets but all tucked in tightly, with a bed cover spread over them. He did not know that he should turn back the blankets and get into the bed. The bed cover was loose; he was sleeping under that linen cover with his overcoat and shawl. I showed him how to get under the blankets. He was extremely touched". Ramanujan himself wrote to M. Narasimha Iyengar who was coached by him in Mathematics for Intermediate Examination that the first few days in Cambridge were not every congenial because he was not communicative enough with other students in the University and as that brought him some little annoyance from the students. Later, however on the intervention of University authorities, the students began to treat him with great respect.

Ramanujan wrote to Krishna Rao about two months after his arrival in Cambridge "Till now I did not feel comfortable and I would often think *why I had come here.* It is due to difficulty of getting proper food. Had it not been for the good milk and fruits here, I would have suffered more. Now I have determined to cook one or two things myself and have written to my native place to send some necessary things for it...Now I am living in the College and going to stay here for the future also even though it is more costly than lodging houses, as it will be inconvenient for the professors and myself if I stay outside the College". The College also had provided him with necessary facilities. His room was electrified and had a gas stove also. He cooked his food and the menu generally consisted of rice, vegetables, *sambhar* (a concoction of some vegetables and pulses) and butter-milk. Janakiammal remembers that he did not know cooking while he was in India. But in England he had to do it by himself. His favourite item was *pongal* (boiled green gram and rice). Dr. C.D. Deshmukh who later held very important positions in India as Finance Minister

Central Government, Chairman University Grants Commission and Vice-Chancellor Delhi University came to Cambridge in 1913 remembered the taste of *rasam* (a thick spicy beverage) Ramanujan would cook and entertain the Indian students amongst whom he was very popular. For that matter he had many friends who were not Indian. Only *vegetarianism was an article of faith for him.* This is borne out by an eposide that took place in London during the last year of World War I. He came to London and paid a casual visit to a Lodging House where many Indian students resided. The landlady served breakfast assuring him that it was pure vegetarian. But the beverage given was ovaltine. When out of curiousity he looked at the tin, he was "horrified" to note that it contained a little powdered egg also. An upset Ramanujan immediately left the lodge and wanted to catch the first train for Cambridge. As he approached the railway station, there was heavy bombardment of the neighbourhood by German bomber planes and he could reach Cambridge with great difficulty. Later he worte to the landlady about the entire happening and added for the sin of *taking something non-vegetarian God punished him with the air-raid.*

Besides the problems of food and adjustment in an alien environment, Ramanujan had another worry: the work for which he had come to Cambridge was not proceeding according to his original plan. He came in April and War broke out in August. He wrote to Krishna Rao "I am very slowly publishing my results due to the present war. A lecturer here whom I know well and from whom I received some help to publish my result has gone to war. The other professors *have lost their interest in mathematics owing to the present war.* One of the professors here some days back remarked that I have come to England in the most unfortunate time.... I have changed my plan of publishing my results. I am *not going to publish any of the old results in my Notebook till the war is over.*" Ramanujan wrote to his mother on 11th September, 1914, giving more details of the havoc created on land, sea and in the air by the war which affected "crores of people. It is not one or two crore". But he felt that she must have been worried about her son, so he assured her "There is no war in this country. War is going on in a neighbouring country" which is as far

from here as Rangoon is from Madras. Further, he told her that he was not alone. "Lakhs of persons have come here from our country to join the forces. Seven hundred rajas have come from our country to wage war. Ultimate victory will come to the king of this country." However, in his letter, he did not mention that two days after the war broke out Gandhi had reached London and a delegation of the Indian National Congress was already present to affirm India's active participation. Gandhi had exhorted Indians' "to think imperially"and Balgangadhar Tilak who had recently been released from Mandalay Jail in Burma, declared "it is the duty of every Indian to support His Majesty's Government to the best of his ability". The national leadership of India had not become super-agent for supplying mercenaries–, they genuinely believed that this was the surest way to *Swaraj* or Home Rule for the country. Gandhi asserted in a letter to Sir Surendranath Banerjee "I implicitly believe that if we were to devote our attention *exclusively to recruiting we would gain full responsible government in a year's time if not sooner*". And he did actually devote his attention eclusively to recruiting. In Kheda district of Gujarat, for example, he gave a call for "twenty recruits from every village as sacrifice for Empire and Swaraj". Due to his activity being so intense Gandhi earned the title "Recruiting Seargent" from his biographer Tendulkar. Tilak also was not far behind, he was trying to recruit 5,000 able-bodied persons from Maharashtra for the British-Indian Army. All this supplemented the Government's own efforts and a large number of Indians joined, with all sorts of personal and national hopes. But what they got at the end of the war, is a different story—promulgation of Rowlatt Act (which made even *intention* to publish or circulate seditious material punishable by imprisionment) and Jallianwala Bagh (April 6, 1919) Massacre (379 massacred, 1200 left uncared for on the spot).

Ramanujan could not be unaffected by the happenings at home and the attitude in Britain, which became very obvious during his illness in 1917–1919. It would be wrong to assume that he was non-political, and he had no interest outside Mathematics. If Hardy is to be believed Ramanujan was "an ardent politician, of

a *pacifist and ultra-radical type*". It can be said his attitude towards war could not be different from Gandhi's strategy. However the atmosphere at Cambridge was very confusing. As C.P. Show writes "Hardy like Russell and many of the high Cambridge intelligentsia did not believe that the war should have been fought. Further, with his ingrained distrust of English politicians, he (Hardy) thought the balance of wrong was on the English side. He could not find satisfactory basis for conscientious objection ; his intellectual rigour was too strong for that. In fact, he volunteered for service under the Derby Scheme, and was rejected on medical grounds. But he felt increasingly isolated in Trinity, much of which was *vociferously bellicose*. Russell was dismissed from his lecturership in circumstances of overheated complexity.... Hardy's close friends were away at war. Littlewood was doing ballastics as a Second Lieutenant in the Royal artillery. Owing to his cheerful indifference he had the distinction of remaining a Second Lieutenant through the four years of war. Their collaboration was interfered with, though not entirely stopped. It was the work of Ramanujan which was *Hardy's solace during the bitter college quarrels*".

However Ramanujan was not sitting idle. On Hardy's advice he was attending lectures not only at Trinity but also at other colleges of Cambridge. Hardy realised the need of acquainting him with new methods. He wrote "I had to try to teach him and in some measure I succeeded, though obviously I learnt from him much more than he learnt from me. He was never a mathematician of modern school and it was *hardly desirable* that he should become one" so he decided to work with Ramanujan in a *participative way*. "The limitations of his knowledge were as startling as its profundity. Here was a man who could work out Modular Equations and theorems of complex Multiplication to orders unheard of, whose mastery of Continued Fractions was on the formal side at any rate beyond that of any mathematician in the world, who had himself found the functional equations, the Zeta Function, and the dominant term of many of the most famous problems in the Analytic Theory of Numbers; and he had *never heard* of Doubly Periodic Functions or of Cauchy Theorems and

had vaguest idea of what the Function of a Complex Variable was".

Ramanujan himself was responding. He wrote to Krishna Rao on 13th November 1914 "I have changed my plan of publishing my results. I am not going to publish any of the results in my note book till the war is over. After coming here I have learned some of their methods. I am trying to get new results by their methods so that I can easily publish these results without delay. In a week or so I am going to send a long paper to the London Mathematical Society. The results in this paper have nothing to do with those of my old results." He confirmed this information in his letter to an old school mate S.M. Subramanian "I am doing my work very slowly. *My notebook is sleeping in a corner* for these four or five months. I am publishing only my present researches as I have not yet proved the results in my notebook rigorously In the same letter he told Subramanian about what he called *a curious function :*

Arrange all the rational numbers thus

$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{1}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{4}{1}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}$, and construct the function

$$F(x) = \frac{\cos(\sec 1/\sqrt{1-x})}{2} + \frac{\cos(\sec 1/\sqrt{1-2x})}{2^2} + \frac{\cos(\sec 1/\sqrt{2-x})}{2^3} + \frac{\cos(\sec 1/\sqrt{1-3x})}{2^4} + \frac{\cos(\sec 1/\sqrt{3-x})}{2^5} + \ldots\ldots$$

Since the cosine is numerically not greater than 1 the series is absolutely convergent. If you give any rational value x one term will be indefinite as (cos (sec ∞) is not definite. Hence F (x) does not exist for all rational values of x. Again there is another curiosity here. Take a rational point say 1/2. If you approach from the left to the point 1/2, the cos (sec $\sqrt{1-2x}$) approaches the form cos (sec $1/\sqrt{+0}$) which is indefinite. But if you approach the

point 1/2 from the right, then cos(sec1/$\sqrt{1-2x}$ approaches the form

$$\cos(\sec 1/\sqrt{-0}) = \cos(\operatorname{sech} 1/\sqrt{+0}) = \cos 0 = 1.$$

Similarly for every rational point. Hence F (x) exists if you approach the rational points from the left side; and F(x) is obviously finite and indeterminate when you approach the rational point x from the left side ; and F (x) is obviously finite and determinate for every irrational value of x. Just imagine the function remembering that

(i) between any 2 rational points there are an infinite number of points ;

(ii) between any 2 irrational points there are an infinity of rational points ;

(iii) every irrational point is the limiting point of some sets of rational points as well as some other sets of irrational points ;

(iv) every rational point is also the limiting point of some sets of rational points as well as some other sets of irrational points."

C.47

He concluded the letter adding "Now as well as in future I am not in need of anything as I have gained perfect control over my taste and can live on mere rice with a little salt and lemon juice for an indefinite time."

As the above letter indicates Ramunujan had really started using what he had called new methods. His tutor Dr. E.W. Barnes who is known in mathematical literature for the Mellin-Barnes contour integration method, submitted his Report to the Registrar of Madras University on 8th November, 1915 noting that "The work and progress of Mr. S. Ramanujan is excellent. He is entirely justifying the hopes entertained when he came here. There can be no doubt at all that his scholarship should be extended until, as I confidentially expect he is elected to a Fellowship at the College. Such an election I should expect in October 1917." Three days later Hardy also sent his Assessment to Registrar Dewsbury: "Ramanujan has been much handicapped by the

war. Mr. Littlewood who could naturally have shared his teaching with me has been away, and one teacher is not enough for so fertile a pupil...He is beyond question the best Indian mathematician of modern times. He will always be rather eccentric in the choice of subjects and methods of dealing with them. But of his extraordinary gifts there can be no question, in some way he is the *most remarkable mathematician* I have ever known".

Ramanujan himself was keeping his people well-informed about his work. Within six months of his stay at Cambridge he had sent some pre-prints of his papers to Ramachandra Rao and Seshu Iyer. On the day Hardy sent his Assessment to Madras University, Ramanujan also sent a paper to S. Narayana Iyer that was read before the London Mathematical Society. In his letter he wrote "As for your suggestion to publish more important and general results, I have *first to read French and German works and journal* and to become familiar with extremely rigorous proofs and then publish my results. Unless I remain here for two years more I cannot do all I have to do. If I just consider the enormous losses to this College owing to this terrible war, I think I have no voice to ask the authorities of this College to do anything if they cannot afford to retain me here of their own accord. There were about 300 students and even more than that sometimes before the war, but there are only about 150 this year. Perhaps there may be very few only next year. If I could remain here for more than one year I should like to go over there for the coming long vacation so that I may not disappoint my people to whom I promised to return at the end of two years of my stay here."

In the letter he requested Narayana Iyer to tell Sir Francis about his work in Cambridge, who would make all the necessary arrangements for further stay at Trinity and also a trip to India. Accordingly on December 15,1915, Spring wrote to the Registrar of Madras University :

"the accompanying papers refer to Mr. Ramanujan, in whom I am interested if for no other reason than that he was *discovered while working as a clerk* in my office. The papers have reached me from my office Manager, Mr. S. Narayana Iyer, M. A., L. T. who

himself a good mathematician was chiefly instrumental in bringing Mr. Ramanujan into notice. In the *interest of India's reputation as the birthplace of some good mathematicians* of which Mr. Ramanujan is, I am assured, one of the most remarkable, I hope it may be possible for the University to insure the uninterrupted continuance of his studies, but chiefly of his original research work at Cambridge, by promising him the enjoyment of his present scholarship upto the end of March 1918. The Syndicate will, I have no doubt, give the matter best consideration".

With the Report of Barnes, Assessment of Hardy and Spring's recommendation before the Syndicate, there was no hesitation in granting Ramanujan extension of scholarship for one more year. This allowed him more time to concentrate on his work. Trinity College also recognised his original research and on that basis awarded him the degree of Bachelor of Arts in March 1916.

Ranganathan has called 1914–1918 as the "second period of superactivity for Ramanujan. If the number of publications is the criterion, this characterisation is perfectly correct. While he was still in India Ramanujan had published only five papers, between 1911 and 1913, apart from contributing some problems and brief notes to the *Journal of Indian Mathematical Society*. But at Cambridge upto to December—end, 1916 as many as thirteen papers had appeared in various research journals :

Quarterly Journal of Mathematics

1. "*Modular Equations and approximation to* π" 45(1914), 353—372.
2. "*New expressions for Riemann' function*" $\zeta(s)$ *and* $\equiv(t)$. 46(1915) 250–260.

Journal Indian Mathematical Society

1. "*On the integral* $\int_0^\infty \frac{\arctan t}{t}\, dt$." 7 (1915) 93–96.
2. "*On the number of divisiors of a number*" 7 (1915) 131–133.
3. "*On the sum of the square roots of the first n natural numbers.*" 7(1915) 173–175.

4. *"On the product* $\prod_{n=0}^{\infty}\left[1+\left(\frac{x}{a+nd}\right)^2\right]$" 7(1915) 209-211.

Messenger of Mathematics

1. *"Some definite integrals"* 44(1915) 10–18
2. *Some definite integrals connected with Gauss's sums.* 44(1915) 75–85.
3. *"Summation of a certain series."* 44(1915) 157–160.
4. *"On certain infinite series."* 45(1916) 11 – 15.
5. *"Some formulae in the analytic theory of numbers.* 45(1916) 81–84.

Proceedings of the London Mathematical Society

1. *Highly composite numbers.* 2,14(1915), 347-409.

Transactions of the Cambridge Philosophical Society

1. *On certain arithmetical functions.* 22, No. 9(1916) 159–184.

The last paper, that was read at a meeting of Cambridge Philosophical Society on 25th October, 1915, indicates the new topics and the style that Ramanujan was adopting. This can be seen in the introductory section of paper itself : Let $\sigma_s(o)=\frac{1}{2}\xi(-s)$, where $\xi(s)$ is the Rieman-Zeta function. Further let

$$\sum_{r,s}(n)=\sigma_r(0)\sigma_s(n)+\sigma_r(1)\sigma_s(n-1)+\ldots+\sigma_r(n)\sigma_s(0) \qquad (1)$$

In this paper I prove that

$$\sum_{r,s}\sigma(n)=\frac{\Gamma(r+1)\,\Gamma(s+1)\xi(r+1)\,\xi(s+1)}{\Gamma(r+s+1)\,\xi(r+s+2)}\sigma_{r+s+1}(n)+$$

$$+\frac{\xi(1-r)+\xi(1-s)}{r+s}n\sigma_{r+s-1}(n)0\left(\frac{2}{n^{3}}(r+s+1)\right)(2)$$

where r and s are positive and odd integers. I also prove that there is no error term on the right hand side of (2) in the following nine cases :

r=1, s=1; r=1, s=3; r=1, s=5; r=1, s=7; r=1, s=11; r=3, s=3; r=3, s=5, r=3, s=9; r=5, s=7.

That is to say $\Sigma(n)$ has a finite expression in terms of $\sigma_{r+s+1}(n)$ and $\sigma^{r,s,}_{r+s-1}(n)$ in these nine cases, but for other values of r and s it involves other arithmetical functions as well.

It *appears probable from the empirical results* I obtain in §§ 18–23 that the error term on the right hand side of (2) is of the form

$$O\left(n^{\frac{1}{2}(r+s+1+\epsilon)}\right) \tag{3}$$

where ϵ is any positive number, and not of the form

$$o\left(n^{\frac{1}{2}(r+s+1)}\right) \tag{4}$$

But *all I can prove vigorously* is

(i) that the error is of the form $O\left(n^{\frac{2}{3}(r+s+1)}\right)$ in all cases

(ii) that it is of the form

$$O\left(n^{\frac{2}{3}(r+s+3/4)}\right) \tag{5}$$

if $r+s$ is of the form $6m$,

(iii) that is of the form $O\left(n^{\frac{2}{3}r+s+\frac{1}{2})}\right)$ (6)

if $r+s$ is of the form $6m+4$

and

(iv) that is not of the form :

$$o\left(n^{\frac{1}{2}(r+s)}\right) \tag{7}$$

It follows from (2) that if r and s are positive odd integers, then

$$\Sigma_{r,s}(n) \sim \frac{\Gamma(r+1)\ \Gamma(s+1)}{\Gamma(r+s+2)} - \frac{\xi(r+1)\ \xi(s+1)}{\xi(r+s+2)}\ \sigma_{r+s+1}(n) \tag{8}$$

So, when there was *no proper diagnosis* how could *any proper drugs* be prescribed, in particular at a time when no drugs had been found for tuberculosis. In such a situation how could any sanitorium attach any importance to Ramanujan's diet, especially when his palate was hard to please.

Nothing is known about the diet that was prescribed for Ramanujan by the doctors at the various sanitoria. Although Hardy used to meet him frequently and discuss mathematics, yet by nature he could not be expected to note such 'trivialities'. It was only Matlock which was visited by Ramalingam who found the "shocking neglect" of Ramanujan's dietry requirements. Although Subbaraya Ramalingam of Caddalur (a town 205 kilometers to the south of Madras) had joined the King's College of London University for B.Sc. Engineering degree in 1910, he was there at the harbour in 1914 to welcome Ramanujan. During his stay in Britain he had to do war duty also and finally he joined the Indian Engineering Service. While Ramanujan was at Matlock House, he found out his whereabouts from Hardy and after exchange of telegrams visited the sanitorium on Sunday June 16, 1918 and stayed there till Tuesday 18th June spending about 50 hours with Ramanujan. The Matron allowed him to take tea with Ramanujan in his room but he had to take other meals in the dining Hall. Later on he wrote a long letter to Hardy on 23rd June, 1918 (quoted in full by Rankin in his article entitled 'Ramanujan as a patient'; *Proceedings, Indian Academy of Science* (Mathematics Section) Vol. 93 Nos. 2 and 3 Dec 1984, pp. 79-100). saying "I was shocked and horrified to find him in the thin, weak and ematiated condition... I was conversing with Ramanujan all sorts of topics, personal, political, war, Indian social, Christian mission etc. I had not the slightest reason to suspect or think that he was *going off* at all as one or two were presumptuous enough to think..."

Then he gave details of the meals that were served to Ramanujan :

"For *breakfast* he is having *scrambled eggs on toast* and tea, well, this can hardly be improved perhaps, though I can make

something better to suit Ramanujan's palate, but still for his health's sake, no improvement is desirable in this meal. For *Lunch* he was having plain boiled rice, chillies and mustard fried in butter, lately began having cucumber and lemon fruit. The doctor was kind enough to provide for a day or two some green peas specially for him. I maintain this meal could very considerably be improved. *Tea,* more or less repetition of breakfast with green onions substituted for eggs at times. *Dinner* more or less a repetition of lunch+a glass of milk. I believe a glass of milk is given about lunch time as well.

"I asked the doctor if he recommended or prohibited any particular items of diet e.g. *whether he objected to any pungent stuff like curry*. The doctor advised that Ramanujan could take anything he liked best.

"Ramanujan said 'he did not like porridge nor did he care for bread and milk *on which diet he had been formerly for some months*'I asked him why he did not think of going to a vegetarian sanitorium. His reply was 'India Office was unable to recommend any vegetarian sanitorium."

Ramalingam further mentions in his letter that Dr. Labbhu Ram, who was associated with Matlock House was also to some extent responsible for depressing Ramanujan as he once said "As long as you are a patient and not well, you are not free and the doctor has control over your movements". Also from the doctor, Ramanujan got the impression that he would have to stay at Matlock House for at least twelve months. So Ramalingam suggested that it would be better for Ramanujan to shift elsewhere, but the latter felt that it would be advisable to consult Dr. Labbhu Ram first.

Ramalingam had told Hardy at the outset in his letter "Well it is with regard to food that I have to write somewhat harshly and tersely and at a good length". So apart from giving details of meals served he devoted a few paragraphs which indicated the attitude of the Matron and the kitchen staff towards Ramanujan. He wrote

"I called on the Matron and consulted her if I myself could cook something for Ramanujan. "*No you can't go to kitchen*' was the answer. Then I offered to write a recipe and asked them if they could cook the recipe. Then there was some trouble about the old cook leaving. I was rather keen on having my stuff and recipe tried while I was there, so that I could see it was properly cooked and also if Ramanujan liked it. Well the opportunity was really denied to me till Tuesday 1 p.m. (the time when Ramalingam left Matlock).

"On Monday there was prepared some mixture of macaroni custard and something also for pudding. Ramanujan did not get any pudding and on enquiry I found—this was before I went to lunch—that it was macaroni and something else. I at once requested that Ramanujan should be served with whatever pudding that is going on in the dining room and it must be left to Ramanujan to eat it or leave it. Ramanujan afterwards told me that *he liked macaroni prepared with cheese.* It seemed unfortunately that the *sanitorium people couldn't get cheese.* Well goodness, I thought that the sanitorium had a priority claim for such foodstuffs. Anyway... I managed to spare some for Ramanujan and send onto him about one lb, of cheese on Friday afternoon (21st. inst.).

"Also, I understood from Ramanujan that the Matlock sanitorium people couldn't get bananas for him. Whereas on Tuesday afternoon I saw a fellow traveller from Sheffield who was in the panel or *ward of Dr Ram (Labbhu Ram) carrying a bunch* of about a dozen bananas On an enquiry I gathered they cost 4 d each. Still they are obtainable.

"Nextly Ramanujan complained that he used to get once e.g. *when Dr. Ram visited him delicious rice pudding, sago pudding* etc. and that he was not getting them lately. Also I understood from Ramanujan that the sanitorium people *couldn't afford butter to fry his potatoes in.*

"Therfore you see, Prof. Hardy, that there is plenty of scope for improving his diet and *I saw little chance of improving it so long as he was in that sanitorium*".

Then Ramalingam describes the discussion that he had with Dr. Labbhu Ram and informs Hardy "Dr. Ram also gave me some hope of Ramanujan's life and added that *food alone could save him.* Dr. Ram asserts that there are *very few doctors who would care to have Ramanujan in their nursing home* and fewer still who would *bother with humouring Ramanujan's palate...* Dr. Ram thinks that Dr. Kincaid is the only kind and charitable doctor that will tolerate Ramanujan".

Ramalingam concluded the long letter discussing various possibilities of shifting Ramanujan to some other place where he could get better care and treatment and also offered to drive him personally. He also enclosed a letter for Ramanujan and requested Hardy to "please peruse it and post it straight away" A letter that Ramanujan had written to Ramalingam on June 19 (one day after the latter had left Matlock) was also appended, in which the former had described further deterioration in food. He wrote "The old cook has left this place day before yesterday. The present cook spoiled yesterday all *appalappu* (small papadam) by scorching some and leaving some raw. The curried rice was like *akshata* (uncooked rice used in rituals mixed with turmeric) and as hard as uncooked rice. Yesterday, I had no dinner. At least I had some breakfast and plain boiled rice before. Even these she can't prepare properly now. And so please don't send me anything except *avarakkai* (broad beans), *vendakkai* (or ladies finger) and *nillakadalaiurundi* (ball of sweetened groundnut kernel. *Don't send anything to the Matron with instructions or any such thing*".

It is not known what action was taken by Hardy on Ramalingam's letter. Did he discuss the problems raised with the Matlock Superintendent, Dr. Kincaid ? Did he post the letter to Ramanujan in which he was advised "to eat or drink plenty of porridge, eat tomatoes, bananas, cheese, cream, macaroni; etc. "Bananas and cream go together lovely" ? And if it was posted what was Ramanujan's reaction ?. Or was it Hardy who arranged for Ramanujan's transfer from Matlock to Colinette and Fitzroy ? These and many other questions remain, and possibly will remain, unanswered but Rankin's article giving the unabridged letter of

Ramalingam brings out clearly that the cause of Ramanujan's deteriorating health was not his bigotry—the major responsibility lay elsewhere.

It will not be out of place to mention here an article entitled "Ramanujan in England : An East West Collaboration in Science" that appeared in *The Psychoanalytic Review* vol. 66 No. 3, 1979 pp. 423–442. The article is a result of Ashis Nandy's researches funded by the Indian Council of Social Sciences Research undertaken at the Department of Anthropology and the Centre for Asian Studies of the University of Texas at Austin (U.S.A.), besides Cambridge where he had the opportunity to interview J.E. Littlewood and C.P. Snow and also to consult the relevant papers. This is what he has to say about Ramanujan's life at the sanitoria :

"In May 1917, while still in England Ramanujan had an attack of tuberculosis. He had probably neglected the early symptoms because it *became obvious after the diagnosis* that the illness was in advanced stage and given his Brahamanic food habits and the existing state of medical knowledge incurable. At the most *drugs, nursing and change of climate* could buy him some time. Even this seemed unlikely because he was unwilling to live in a *suitable environment under proper treatment.* He placed more reliance on a *kavacha* (a magical guard plate) to protect himself. The illness also brought out more clearly Ramanujan's finicky attitude towards western food and the distinctive pattern of his orality. His *correspondence with some Indian friends*, letters that he never expected to be shared with others, show that it was not merely a matter of abjuring animal proteins or inability to adjust to English cooking. It was not even austerity, Ramanujan apparently enjoyed good Indian food, whereas he would *not eat even vegetarian Western food, such as cheese, bread, butter or jam* (ispite of his misleading name in Cambridge "Dear Jam")".

One would have thought that in Cambridge Nandy had access to all papers concerning Ramanujan's stay in Britain, but he happened to miss Ramalingam's letter to Hardy dated June 23, 1918. However, it is clear from the Note (pages 441) that he

did study that letter and found it necessary to infer that "Ramanujan would eat plain boiled rice sprinkled with red peper and pickles, rather than any English vegetarian food". but on the other hand Nandy *found it more convenient to omit* important passages from the letter that Ramanujan was taking *scrambled eggs on toast* for breakfast and afternoon tea, loved *macaroni and cheese*, got his appalappu fried in *butter*. And when did any doctor know and Ramanujan was told that he was suffering from tuberculosis ? Did not the confusion last till the end of his stay—nothing serious ? gastric ulcer ? tuberculosis ? some obscure oriental germ infection due to removal of some malignant material before coming to Cambridge ? It would be surprising if anyone doing research on Hardy' relationship with Ramanujan did not care to see the letter that Hardy wrote to Thomson on the eve of Royal Society meeting. In any case this one-sided presentation or deliberate omission is very dangerous methodology of research. After all vegetarianism was an article of faith for Ramanujan, who at one time believed that taking ovaltine was to invite God's wrath. Was it not an extremely bold step on his part to take eggs regularly ? What was expected of him ? To repudiate his origin ? But would that change the attitude of the doctors and matrons ? Anyway now that Professor Rankin has revealed valuable facts in his article, will Nandy revise his pronouncement ? But possibly he may not be able to do so as the facts will not be amenable to the jargon of cosy psychoanalytic hypothesis that he has tried to fabricate.

It is true that Ramanujan like anyone else had his personal weaknesses' After all his background was different from that of the Indian students who were in Britain when he was there. Most of them came from well-to-do families, they had good University education and were already exposed to the so-called Western way of life while still in India. But Ramanujan was the son of a *bhajanika* and an *amin*, he never had a square meal at home—how could he be expected to pounce upon the rich food of Cambridge or London and change his habits overnight ? He was trying to adjust. But he did want to return to India as early as possible. Fourteen months after his arrival in Cambridge he wrote to Subramanian (3rd June, 1915) "I am thinking of

returning to India in a year and see you all again". Then just four weaks later (1st July) he wrote "I am going to take a research degree at the end on one year (hence) and I am thinking of returning to India next year this month and coming back to England if necessary". Two days after getting the B.A. degree he reiterated his intention of a short trip to India and again wrote to Subramanian on 30th March, 1916 "I am thinking of returning there in June (this year) for the long vocation. 'But when he got more engrossed in his work he had to change his plan. As a Cambridge mathematician has noted "Ramanujan was working thirty hours at a stretch and then sleep for thirty". He wanted to do as much work as possible before going home. But he kept on sending letters home to his parents and the wife, however, poor Janakiammal never got those letters. So in his mother's letters to him there would be no mention of Janaki. This is possibly the reason why he stopped writing to his people. So, Subramanian wrote to Hardy about it. Hardy in his reply of 20th September, 1917 said "It was only a few months ago when he was at Nursing Hospital here that we discovered that he was not writing to his people nor apparently hearing from them. He was very reserved about it and it appeared that there must have been some quarrel. At that time he was really very ill indeed...the prospects as regards Ramanujan's health are now decidedly better....I showed your letter to Ramanujan (who is now in Cambridge) and got him to promise to write to his people. We are most anxious that any trouble which may have arisen should be cleared away". His loneliness became more acute when he was moving from sanitorium to sanitorium. He requested his mother to send Jankiammal with someone to join him. But Komaltammal wrote in reply what he had jokingly told his wife at the time of leaving India "Janaki is young and beautiful. Englishmen will pester her. Your research will be affected and your health will also deteriorate". Presumably this would have been the last straw to break the camel's back, so to say. Dejected and desperate he tried to end his life by falling on the track of the underground railway of Greater London, in front of a coming train. Luckily the driver stopped the train well in time to save his life. As in British Law unsuccessful suicide is a cognisible offence, Ramanujan was, accordingly taken to the

police station for interrogation. Hardy rushed there as soon as he got the news. He pleaded with the police officer that Ramanujan was a Fellow of Royal Society and hence deserved special consideration. Both Hardy and the police official knew that Ramanujan had not yet been elected Fellow of Royal Society and that no F.R.S. was immune from Law. However Ramanujan was let off and the police official said later that he did not want to harass an important personality who had taken that extreme step in desperation and under severe strain. Janakiammal says that Ramanujan never told her about his suicide attempt, but feels that he, might have done that in desperation because she was not with him.

All this trouble, worry, sickness and strain, however, could not stop Ramanujan from producing Mathematics, independently or in collaboration with Hardy. From January 1917 till his return home in 1919 he published 12 papers.

Messenger of Mathematics

1. *A series for Euler's constant r*; 46 (1917) pp. 73-80.

Transactions Cambridge Philosophical Society

1. *On certain trigonometrical sums and their applications in the theory of numbers*; 22 No. 13 (1918) pp. 259-276.

Proceedings Cambridge Philosophical Society

1. *On the expression of a number in the form* $ax^2+by^2+cz^2+dt^2$; 19 (1917) pp. 11-21.
2. *Some properties of p (n), the number of partitions of n*; 19(1919), pp. 207-210.
3. *Proof of certain identities in Combinatory Analysis*; 19(1919); pp. 214-216.

Comptes Rendus

1. *Use formule asymptotique pour le nombre des partitions de n* (with Hardy) 2. Jan. 1917.

Proceedings London Mathematical Society

1. *Asymptotic formulae for the distribution of integers of various types* (with Hardy); Ser 2, 16(1917) pp. 112-132.

2. *Asymptotic formulae in Combinatory Analysis* (with Hardy); Sec. 2, 17(1918) pp. 75-115.

Quarterly Journal of Mathematics

1. *The normal number of prime factors of a number n* (with Hardy); 48(1917) pp. 76-92.

Proceedings Royal Society

1. *On the coefficients in the expansions of certain modular functions* (with Hardy) (A), 95(1918) pp. 144-155.

Journal Indian Mathematical Society

1. *Some definite integrals* 11 (1919) pp. 81-87.
2. *A proof of Bertrand's postulate* 11 (1919) pp. 181-182.

Besides the published papers, Ramanujan left a mass of unpublished results. Hardy recalled "All of Ramanujan's manuscripts passed through my hands, and I edited them very carefully for publication. The earlier ones I rewrote completely. I had no share of any kind in the results, except of course when I was actually a collaborator, or when explicit acknowledgement is made. Ramanujan was almost absurdly scrupulous in his desire to acknowledge the slightest help".

9

Goodness Gracious ! Black Diamond ! ;

There is a folk saying popular among the pesants of Eastern Uttar Pradesh :

> *Jaey dhān, rahe kyārī kabhī to āeygī kisān ki bārī*
> (let the paddy go, but the field remain the farmer's turn will come sometime, after all)

For Ramanujan the 'field' was his mathematics and the 'paddy' his recognition by the mathematical community. Many a time he failed in his attempts and now he had the same experience at Cambridge also. After graduating Bachelor of Arts by research, he put up his candidature for Fellowship of Trinity College which if awarded would have given him unrestricted opportunity to continue his research. But the decision-making bodies of Trinity College, found him unfit for the honour. Was it a continuation of the tradition-first rejection then realisation of the mistake and pilling up honour after honour as was the case with Dewan Bahadur R. Ramachandra Rao and the suspecting Hardy, or was it the response similar to that of Baker and Hobson or, by any chance a case of 'pinching the donkey's ears, when helpless before the washerman' (*dhobi se n jitain to gadhe ka kan ainthain*) because Hardy, as C.P. Snow points out was isolated in the College quarrels;' but was too strong to be harmed

and untrained mathematical worker, Srinivasa Ramanujan was only his protege ?

Then Ramanujan looked a little higher towards which no compatriot had ventured to look. He put up his candidature for Fellowship of Royal Society. The Royal Society was the most prestigious scientific body in the British Empire. It was founded in 1660 and the Charter of Incorporation was issued two years later by King James II. Amongst the presidents of the society were such distinguished scientists as Issac Newton, Humphry Davy, T.H. Huxley. At that time the famous physicist J.J. Thomson was the president. It is not known if anyone had any queries about Ramanujan's mathematical excellence, but it appears that serious questions about his health were raised. This prompted the President to enquire from Hardy the actual state of Ramanujan's health. In his reply Hardy said :

"I should like to say more clearly what the position about Ramanujan is. I wouldn't have done so if you hadn't asked me. I am much afraid of prejudicing him by persistent pushing in view of his *double candidature* for a Fellowship and Royal Society. If he had not been ill I would have deferred putting him up a year or so : not that there is any question of the strength of his claim, but merely to let things take their ordinary course. As it is I felt no time must be lost. I hardly hoped that he would have much chance this year. I don't know whether he has or not. If he has, no doubt his illness will have much to do with it; and I should tell you precisely how (so far as I can know) matters stand."

(Then Hardy gives details of the various doctors'—Shaw Kincaid and others—conflicting views about the nature of Ramanujan's illness which have been given in the earlier chapter). The letter further says :

"For my part, I think *probably* he will be alive in a year's time, and that he may recover completely. If he were like Jeans—I mean who made a scientific fight to get right he would have a better chance. Like all Indians he is fatalistic and it is

terribly hard to get him take care of himself Everyone too, is frightened of the continual illness and solitude affecting his mind.

"As I said, I am nervous about trying to rush him, and I am aware that for the time being, I am not an ideal supporter. And I realise that the Royal Society has many other things to consider. But there is no doubt that (*especially after his disappointment in the Fellowship*) any *striking recognition* now might be a tremendous thing for him. It would make him feel that he was a success, and that it was worthwhile going on trying. It is this much more than the *fear of the Royal Society losing him entirely which seems to me important.*"

"I write on the hypothesis that his claims are such as, in the long run, in any case, could not be denied. This is to me quite obvious. There is an absolute *gulf* between him, and all other mathematical candidates." Later the Council of Royal Society met on February 14 and 21 to prepare a provisional list of candidates to be elected Fellows. As the letter of Hardy to Thomson was in reply to a query from the latter about Remanujan's health and the letter is without date it is most likely that Ramanujan's candidature was considered at the first meeting itself and someone raised problems about his health. Hardy's letter cleared all doubts and his name was included in the list at the second meeting. The provisional list was read at the Council's meeting of February 28 and approved. Then according to the regulations of the Society the list was published on March 7, 1918 in the *Nature*. The actual election took place on May 2 and the candidates became Fellows from that date. But it appears that as soon as Ramanujan's name was included in the list of 28th February, it was thought that the time consuming process was only a formality so some biographers, like Ranganathan and P.K. Srinivasan have taken 28th February as the date of Ramanujan's admission to Royal Society, while Hardy seems to have cabled to Dewsbury when the list was to appear in the Nature. Accordingly Madras University sent the press handout on March 6, 1918.

When the news reached India, anyone who knew or had heard about Ramanujan, felt proud of him—here was an Indian

who had struggled through his way to be admitted to the prestigeous club of scientists. The Registrar of Madras University received a cable from Professor Hardy and a press-note was sent on March 6 by the University to all important dailies announcing the good news of the first Indian's election as a Fellow of Royal Society. A felicitation meeting was held in the Big English Lecture Hall of Presidency College, Madras at the initiative of the former Advocate General, S. Srinivasa Iyengar. Professor K.B. Madhava, who only a few years ago had hesitated to lend his books to Ramanujan doubting the youngman's ability, spoke quoting anecdotes about Fellows from de Morgan's book and play upon the initials F.R. and S. The Government of Madras also did not lag behind. On April 11, further appropriation from the University vacation Lecturer's Fund for stay at Cambridge was sanctioned.

Election of Ramanujan as a Fellow of Royal Society was a matter of happiness for the Indian Mathematical Society also. After all, the Indian Society had published his questions and short notes at a time when he was not even a petty clerk. Also the founder members and office-holders like Ramachandra Rao, Ramaswami Iyer, Narayana Iyer had a hand in grooming and helping him reach that stage. He was mentioned at the First Conference of Indian Mathematical Society held on December 27, 1916 Lord Pentland, the then Governor of Madras who had sanctioned a sum of Rs. 10,000 for Ramanujan's visit to Cambridge said "It (Mathematics) is at any rate, a science which has always made a strong appeal to the Indian intellect and has received from it important contribution. It is for example, I suppose probable that to India we owe the decimal notation and the invention of algebra, and India can point at least one epoch in which it produced a number of brilliant mathematicians and astronomers. Is there any reason why such an epoch should not recur. None that I can see. At the present time a young Indian student Mr. S. Ramanujan is studying at Cambridge whose career we in South India are *watching with keen interest and high anticipation.*"

Two years later when the second session was held at Bombay in January 1919, the news of Ramanujan's Fellowship of Royal

Society had already reached India. A resolution congratulating Ramanujan was passed. Ranganathan recalls that the announcement was made in the morning session when Prof. R.P. Paranjpye-Principal of Fergusson College, Pune and later Vice Chancellor, University of Lucknow, was in the chair and Ramaswami Iyer, whose help Ramanujan had sought in 1910 and in whose house he had stayed on the way to Namakkal for seeking Namagirithayar's permission "went into ecstasies."

After the *striking recognition*, Ramanujan put up his candidature for Fellowship of Trinity College once again. He knew that Professor Thomson, had become the Master of Trinity College who knew all about him. But still he suspected that questions about his health may be raised when he appeared for the preliminary interview. As the Trinity College regulations specifically barred the insane from being elected to its fellowship and his attempted suicide could have been construed as a sign of insanity. so as a matter of precaution he carried in his pocket a medical certificate saying he was mentally sane. Luckily no one raised any objection to his candidature after being found deserving enough to become Fellow of Royal Society and Ramanujan became a Fellow of Trinity College on October 13, 1918. The whole story was narrated by S. Chandrasekhar at a function in 1968 when he received the "Srinivasa Ramanujan Medal" of the National Institute of Sciences (now Indian National Science Academy).

Hardy lost no time and the very next day (14th October, 1918) wrote to Dewsbury, the Registrar of Madras University that Ramanujan as a Fellow of Trinity College was entitle to an honorarium of £250 a year for six years and the Fellowship involved no duties nor any obligations to residence. He also suggested to Madras University, that a Chair in the University be created specifically for Ramanujan. In the meantime several Indian friends wrote letters asking him to return. It is not known what and how they wrote, but Hardy got the impression that "the suggestions are not made in the most tactful way and it seemed to have turned Ramanujan against the idea of going." What a quandry poor Ramanujan was in—, miserable life at the hands of

doctors and matrons of sanitoria and the future life of uncertainty in India to which his enthusiastic friends were inviting.

Six weeks later Hardy wrote another letter to Dewsbury emphasing the need of Ramanujan returning to India with provision to visit Cambridge whenever he liked. He said.

"I have been meaning for some days to write to you again about Ramanujan... I think it is now time that the question of his temporary return to India and of his future, generally, should be reconsidered. There is, at last, I am profoundly glad to say a quite definite change for the better ... The consensus of medical opinion is that, he has *been suffering from some obscure partially diagnosed source of blood poisoning which has now dried up* and that it is reasonable to expect him to recover his health completely and if all goes well fairly rapidly. He would now be almost fit to make the journey if accompanied by a careful friend. However the other reasons which made his continuous stay in England desirable (his candidature for the Royal Society for a fellowship) have now ceased to have importance. At various times we have felt considerable anxiety about his *mental state*. I do not think there is really anything seriously amiss with it. But the long illness, and spells of comparative soltitude have undeniably had an effect and he has been subject to *fits of dispersion and been difficult to manage*. This (with a man of his rather nervous temperament and abnormal quickness of mind) is only natural and almost inevitable. But I think that (assuming his physical condition to have been improving as it has lately) a return for a while to his own country would be a very good thing ... But if I were assured officially (as I have been unofficially) that it is the intention of Madras to make permanent provision for him, in a way which will leave him free to do research and to visit England from time to time, I would support the proposal for his return, and no doubt he would be willing to go ... He will return to India with a scientific standing and reputation such as no Indian has enjoyed before, and I am confident that India will regard him as *the treasure he is*. His natural simplicity and modesty has never been affected in the least by success, indeed all that is wanted is to *get him realize that he really is a success*."

Hardy's letter acted as a catalyst and within less than two weeks the Syndicate decided to grant a sum of £250 annually for 5 years beginning April 1, 1917 in *recognition of his services to Mathematics*. A letter was sent to Ramanujan on December 9. The Syndicate could not create a Chair for him at the University, but made sufficient provision for him to work without any serious financial worries. Add to this the amount of £250 from Cambridge—the Fellowship money, he would be very well off. When the letter reached him, he was in the Colinette House Sanitorium. He was reminded of his early days of poverty and realised that now he was himself in a position to help poor students. In his letter to the Registrar (January, 11, 1919) he wrote ".. I feel, however, that, after my return to India, which I expect to happen as soon as arrangements can be made, the total amount of money to which I shall be entitled will be much more than I shall require. I should hope that after my expenses in England have been paid, £50 a year will be paid to my parents and that the surplus, after my necessary expenses are met, should be *used for some Educational purpose*, such in particular as *the reduction of school fees for poor boys and orphans and provision of books in schools*. No doubt it will be possible to make an arrangement about this after my return ... I feel very sorry that I have not been well. I have not been able to do so much mathematics during the last two years as before. I hope that I shall soon be able to do more and will certainly do my best to deserve the help that has been given me."

Hardy also was very pleased at the provision made for Ramanujan by Madras University and in his letter of January 15, 1919 expressed his appreciation. On the other hand some well wishers of Ramanujan on hearing about his serious illness were very keen to have him back in India as early as possible. Dewan Bahadur R. Ramachandra Rao, who was being kept informed about Ramanujan's health by Hardy, wrote to the Registrar about making arrangements for Ramanujan's return in March or April. Littlehailes, who had lent a helping hand in Ramanujan's development soon after attending the Bombay Conference of Indian Mathematical Society had become Director of Public Instruction, Madras Presidency, took steps to create a Chair for Ramanujan. He moved the Syndicate. The Chair was created. But was Ramanujan able to occupy it ? It was too late.

While preparing to return home, Ramanujan kept himself busy investigating the congruence properties of partitions and algebraic relations between certain infinite products. The two papers were presented, strangely enough, the day he was leaving Britain for India. Results on the first topic were published in the *Proceedings of London Mathematical Society*, 2, 18 (1920) and *Mathematische Zeitschrift*, 9 (1921) 147-53 under the same title "Congruence properties of partitions." The other paper also appeared in the same issue of the *Proceedings of the London Mathematical Society*.

Hardy in one of his lectures on the work of Ramanujan later asserted that "very little was known about the properties of the partition function p (n) and Ramanujan was the first and for a long time the only mathematician to discover any such properties and his theorems were discovered in the first instance, by observation. Macmahon had calculated a table of p (n) for the first 200 values of n. Ramanujan observed that the table indicated certain congruence properties of p(n). In particular $p(5m+4)\equiv 0 \pmod 5$, $p(7m+5)\equiv 0 \pmod 7$, and $p(11m+6)\equiv 0 \pmod{11}$. The first two of these formulas he proved using Euler's and Jacobi's identities :

$$\prod_{k=1}^{\infty}(1-x^k)=\sum_{k=-\infty}^{\infty}(-)^k x^{\frac{1}{2}k(3k+1)}$$

and $$\prod_{k=1}^{\infty}(1-x^k)^3=\sum_{k=-\infty}^{\infty}\tfrac{1}{2}(-1)^k(2k+1)x^{\frac{1}{2}k(k+1)}$$

In the same paper he stated without proof two remarkable indentities :

$$p(4)+p(9)x+p(14)x^2+\ldots$$

$$=\frac{5\left[(1-x^5)(1-x^{10})(1-x^{15})\ldots\right]^5}{\left[(1-x)(1-x^2)(1-x)^3\ldots\right]^6}$$

$$\text{and } p(5) + p(12)x + p(17)x^3 + \cdots$$

$$= 7\frac{[(1-x^7)(1-x^{14})(1-x^{21})\ldots]^3}{[(1-x)(1-x^2)(1-x^3)\ldots]^4} +$$

$$+49x\frac{[(1-x7)(1-x^{14})(1-x^{21})\ldots]^7}{[(1-x)(1-x^2)(1-x)^3\ldots]^8}$$

He also proved congruences with moduli 5^2, 7^2 and 11^2 and put forward a general conjecture $24\wedge=1$ that if $\delta = 5^a\ 7^b\ 11^c$ and : mod δ), then

$$p(m\delta+A) = 0(\text{mod } \delta) \text{ for every m.}$$

The conjecture has led to a number of papers which sometimes disprove one or the other. For example : Hansraj Gupta (*Proceedings Indian Academy of Sciences* (A) (1936) 625-629), and Sarvadaman Singh Chawla (*Journal London Mathematical Society* 9 (1934) 2477) have given counter examples. In fact the properties of p (n) still pose a challenge to pure mathematicians.

Ramanujan initiated the process while he was in Britain and finally sailed home. He had landed on British Soil on April 14, 1914 and exactly after 4 years and 333 days (almost half time spent in nursing homes) he set sail for home by S. S. Nagoya on March 13, 1919.

Good Bye to Cambridge !

10

Where is She ?

If you are returning home after a long period of absence how do you feel when you draw nearer and nearer to your destination?

Your heart jumps at every step, your old memories are revived and you start calculating who will be there to receive you when you get down. Faces of near and dear ones appear before your mind and you feel that you are already talking to them.

Ramanujan left Southampton on March 13, 1919 by S. S. Nagoya. Fourteen days on sea once again. Five years ago the had travelled towards a place that he did not know, and now he was returning towards the soil where he had spent twenty seven years of life. But in the meantime he had shot up from obscurity to fame. He must have been thinking all the time that when the ship calls at Bombay, there will be many people to welcome him, at least the ones who he loved most. When the boat berthed in Bombay on 27th March and Ramanujan alighted, he saw only Komalatamal and her brother waiting to receive him. But the one who his eyes were longingly looking for was nowehere to be seen. He did not forget to perform his *śaṣṭāng namaskāram* to the mother but suddenly asked "Where is she ?"

Komalatammal was taken aback and could only ask
"She who ?"
"Your daughter-in-law, Janaki"

"But why do you worry for her ?" asked the
mother in a feeble voice.
"If I do not worry for her who else will."

Ramanujan retorted raising his voice a bit.

Komalatammal realised that her son was no longer her Chinnaswami—he was Janaki's man now. Poor mother had to invent an alibi "Your father-in-law has fallen ill, so Janaki is staying with him."

"I do not know, but she must come immediately". Ramanujan insisted. In fact Komalatammal had not given Janaki any indication of Ramanujan's expected arrival. Now she had no alternative but to send someone post-haste to Rajendram and bring Janaki. Later when Janaki joined him, Ramanujan jokingly told her "Mahatma Gandhi has launched his *satyāgraha* (non-violent struggle) from April 19. He may or may not win, but my *satyāgraha* was successful and you were immediately brought to meet me."

After three days' rest, Ramanujan took Bombay Mail for Madras. When it reached the Central Station on April 2, many friends and admirers were waiting to welcome him. But when he alighted, they were disappointed to find that it was not the Ramanujan they had come to see, but a different person An old family friend, Narasimha Iyengar, who had five years ago persuaded Komalatammal to let Ramanujan travel to Britain and who was the person to have convinced Ramanujan to accept the Madras University offer of £ 250 for stay at Cambridge, in fact had drafted the letter of acceptance and made Ramanujan sign and post it, recalled "When he arrived at Madras Central Station, I met him at the platform but alas I found him not a cheerful, chummy and affectionate Ramanujan that he was but a thoroughly depressed, sullen and cold Ramanujan even after seeing me, a close and affectionate friend of his." Dewan Bahadur Ramachandra Rao, who was the person to initiate Ramanujan's march from obscurity to fame, welcomed him at the platform and later remembered "When I met him alighting from the railway train, I foresaw the end".

Ramanujan first stayed in Madras at Mylapore where his wife also joined alongwith Komalatammal. For more than four months they toured together. Before leaving the Mylapore house, he sent a letter to the Registrar of Madras University on April 23 requesting him to find out from Professor Hardy the total amount of money spent on him during his stay at Cambridge and various sanitoria and also the cost of his voyage, so that the accounts could be cleared. Some time later he sent Hardy a form transferring his whole account to him. Hardy arranged with the bank to settle the debts and place the balance back to Ramanujan's credit. This gesture of Ramanujan was in line with his earlier action in not accepting any dole from Ramachandra Rao as soon as he got a petty temporary job in the Accountant General's Office. He was prepared to accept academic obligation from anyone but not monetary favour. How diametrically opposed this attitude is to the "attitude" that Nandy would us believe "evident in the pathetic applications he wrote in India during 1910-1913, seeking a job and soliciting some attention to his mathematical work" is there for anyone to see.

From Mylapore, Madras, Ramanujan travelled with mother and wife to Namakkal where he stayed only for three days to worship Namagirithayar, although, it appears that he did so only to please his mother. Then in July he came to Kodumudi, a small quiet town on the banks of Kaveri. Here Ramanujan's health began to deteriorate. In August the District Medical Officer suggested that it would be better if Ramanujan was shifted to Coimbatore. Seshu Iyer even made arrangements for his stay and proper treatment at Coimbatore. But Komalatammal said that all her children were in Kombhakonam, so it would be convenient for him also to be there. Ramanujan did not want to displease his mother too much, besides he was very indifferent towards his health so they all moved to Kumbhakonam in September. This was the town where he had spent his childhood, gone to school and college and had numerous friends. They came to see him but all left with a heavy heart, it was painful to see him. When his condition deteriorated further, Dr. Chandrasekhar Iyer himself came from Madras to treat him and only on his insistence Ramanujan moved to Madras for the last time on January 20,

1920 where he stayed at '*Kutia*'. Namperumal Chetty's bungalow in Harrington Road, Chetput, popularly known as Chetput Bangla.

During these months Janaki was taking care of him and even when her mother-in-law was around, there w s no restriction for Janaki in being with Ramanujan. She remembers that when her husband was leaving for Britain, she was not allowed even to talk to him. When they came to Chetput Bangla, Komalatammal stayed on in Kumbhakonam and Jankiammal was joined by her brother P. Srinivasa Iyengar. Both were looking after Ramanujan, Jankiammal says that she had the "good fortune" of serving her husband after his return to India and of looking after his comforts. He was only skin and bones. She would give him food, which mostly consisted of rice, lemon juice, butter and milk at regular intervals and also the medicine prescribed by Dr. Chandrasekhar. He often complained of severe pain in the chest and legs. She kept two vessels for warming water which she used for giving fomentation. Even now the vessels are with her and she says that the vessels remind her of those days.

Ramanujan was nearly all the time lying in bed, which consisted of a mattress and a pillow, spread on the floor. Inspite of severe pain, he was always busy doing mathematics. Janakiammal feels that "this evidently helped him to forget the pain" and remembers "I used to gather the sheets of paper which he filled up. I would also give the slate whenever he asked for it". Janakiammal's brother, R. Srinivasa Iyengar recalled. "With his head propped up on pillows he was continuously working From time to time he transferred his workings from the slate to sheets of paper by his side. And it was known that a vast quantity of papers containing his notes was handed over to the University after his death. We have all heard the word *genius defined as the capacity for taking infinite pains* and it was my special good fortune that it was given to me to watch that capacity functioning concretely before my eyes".

Janakiammal remembers "He was uniformly kind to me. Never complained about any one. Never said anything against

his mother. On the other hand he used to repeat his gratitude to all those who had helped him in his life. One day he confided in me that he would not live beyond thirty-five and asked me to meet the event with courage and fortitude. I began to cry, he consoled me and narrated some funny anecdotes of his life in Britain. One day he told me that *he would not have fallen ill if I accompanied him to Cambridge*". This reminded her that the mother-in law was against the idea. "In fact she had made her son promise that he would not have me for $2\frac{1}{2}$ years. It so happened that once I accompanied my parents to a place near their home, Rajendram for *Navaratra* (a ten day period of austerity in autumn) Mother-in-law on going to Rajendram got suspicious and thought that my parents had sent me abroad to join my husband. She created a scene and my parents somehow pacified her. But the neighbours were very cross with her and chided my parents for not sending me to join my husband".

"My husband was very much attached to his mother earlier, but after his return to India the relationship changed". In fact he had returned home a changed person. he did not think that he was a big man and on Hardy's testimony "his natural simplicity and modesty has never been affected in the least by success; indeed all that is wanted is to get him realize that he really is a success. There was nothing more remarkable in him than his persistent modesty. *Few Indians would not have had their heads a little bit turned by the praise he had received but he was too big a man*". He had become cold in his relationship with his parents, brothers and friends. The only exception was his wife who he was compensating for the neglect over years. He was only preparing to leave the world and because "he had come to India to die". But before he left the world he did not want to put his slate because it was the receptacle of his *dreams*. His friends however, were sad at seeing his plight. Even his childhood friends could not evoke any warmth in him. A typical case is the experience that Anantharaman had. Anantharaman was a younger brother of Ramanujan's class-mate Subramanian. In their childhood they used to play together. Anantharaman was a *Saivite* (belonged to the sect of Siva's devotees) and Ramanujan was a *Srīvaishnavite* (belonging to the sect of Vishnu's devotees). One day Ramanujan

was going to Sarangpani Temple. The *prasādam* (offering distributed among the devotees' used to be very tasty. As it was *Ekādaśi* (eleventh day of the bright phase of the Moon), the *prasadam* was to be specially tasty. Ramanujan took Anantharaman also with him, but to make him appear like a *Srivaishnavite* put *nāmam* and *srīcharnam* on the latter's forehead. But as the markings were fresh, the *pujari* (priest) detected that the boy was a *saivite*, so *prasadam* was not given to him and both the boys returned from the temple laughing at their (especially Ramanujan's) pranks. Then Annantharaman was the same friend with whom Ramanujan had walked a few kilometers even before his wounds had healed after operation of hydrocele, But on return to India when Ramanujan was staying at the Mylapore house, Anantharaman came from Kumbhakonam to see him with a bunch of bananas that were Ramanujan's favourite. "But saya Anantharaman, "He could not even speak and his illness had made him peevish ..... Before he left for England, he was a religious type god-fearing with a devotion to Goddess Namagiri. He used to visit often a small *Anjaneya* temple in Karuppur some two miles east of Kumbhakonam and would on occasions go to Gunasekharam (Elamanur) near Trichi (Tiruchchirappalli) and would advise persons with some mental ailments to go to Gunasekharama for some time. After his return from England, however he had completely changed and it appears that he told my eldest brother Mr. M.M. Ganapathy when mentioned about gods and temples, that it was all foolish talk, and they were only devils".

Ramanujan had changed. But what was the nature of change ? Was he frustrated ? Frustrated in what ? Mathematics ? When he got to a point of success where even intellectual giants would have envied him.? Was it some form of rebellion against the domineering woman called Komalatammal and anything that was associated with her ? Or was it his life in sanitoria, which had deprived him of the will to live as he confessed to his sympathetic doctor Chandrasekhar that he did not want to live any longer.? It would be difficult to characterise the nature of the change unless one is bent upon fitting it into the tight framework of any preconcieved dogma.

11

Vanakkam Chinna

"……out, out, brief candle
life is but a walking shadow"

so speaks Shakespeare through the tongue of Macbeth. Ramanujan had also become a shadow but not walking, simply lying on the floor of Chetput Bangla. And as Janaki's brother noted "had it not been for the bed, nobody could have suspected that there was a man slowly dying". Just a few years ago this body was 5 feet 5 inches tall and stout with a big head and a large forehead, long and dark wavy hair but now it was pale, emaciated, thin and slowly sinking But the glow of the eyes was still there-gradually losing its lusture, his mathematical brain and sense of humour still undimmed. Till about four days before his death he was working on his new-found 'mock-theta founctions' and even in the grim state of his decaying health he had retained his expression. At Kumbhakonam his well-wishers thought he could not get proper medical attention, they wanted him to shift to the distric headquarter Tanjavoor and a person like Ramaswami Iyer was insisting but Ramanujan would not agree. He told Janaki that they wanted to take him to Tanjavoor (pronounced than-sā-voor=one's-dying-place)and he was not ready to go there, but when he was finally induced to move to Madras and came to

Some biographers have thrown too much light on the sour relations between Janakiammal and Komalatammal. For that matter, mother-in-law is not supposed anywhere to be the most lovable person on earth. In certain culture areas like Britain the mother-in law is the *persona-non-grata* for the son-in-law. Innume rable jokes are current and invented everyday depicting her coming to live with her daughter whose poor husband has to suffer the domineering role of the old lady. In certain other culture areas, like India it is mother-in law versus daughter-in-law. These relation-ships are products of the social and economic environment. In her case Janakiammal told the present writer that she did not want to talk about what the mother-in-law did to her. "Now that she is dead she is like a *devi* (goddess) to me. Some writer has raked up this issue and published a number of anecdotes. But I am not the source of information". People close to the Ramanujan household are of the view that it was Komalatammal who had checked the horoscope, fixed the mar-riage inspite of her husband's objection and was very happy with Janaki. Of course petty things were bound to happen in such a household. But when Ramanujan fell ill, the superstitious women thought that her son had fallen ill due to some discrepency between the horoscopes of Ramanujan and Janakiammal. But he had already shot to fame, and as his health was deteriorating day by day, she realised that Ramanujan would soon die and the beneficiary would be his wife And Komalatammal needed some financial support that is why she was trying to keep man and wife apart. Then she was too much a product of her caste and class. When her son crossed the Seven Seas, she feared that he would lose his caste, and if his wife joined him and they had children, her grand-children also would lose their caste. Many other Brahmins had gone abroad about the same time when Ramanujan went, but they had no such worries they were well-to-do and she was poor. A person belonging to the class of the rich can take liberties with impunity and continue to remain a respectable member of his or her caste, but a poor person will have to face the wrath of his or her caste at the slightest violation of the rigid caste-norms.

stay at Chetput Bangla, in the same strain he told his wife now it (death) will be *chat-pat* (=immediate).

At Chetput Bangla he was kept under treatment of Dr. Chandrasekhara Iyer. The best specialist on chest diseases in Madras Dr. Iyer, was not only giving him the maximum personal attention but also would send his daughter with medicine etc to help Janakiammal. When Komalatammal also joined her ailing son, she simply disliked the girl for coming to help Janaki. Ramanujan's well-wishers were trying to do whatever they could to prolong if not save his life. The ex-Advocate General of Madras, Mr. Srinivasa Iyengar was bearing most of the expenses and Raja Namperumal Chetty had given one of houses-Chetput Bangla free for Ramanujan's stay. Members of Madras University Syndicate also tried to help him. But all that was of no avail. Like a candle burning out he was gradually sinking. His friends would come to see him, but he could rarely speak and they would leave with a heavy heart and the sad feeling that he was virually dead, only the skeleton remained.

In the meantime Komalatammal who had unshakable faith in astrology was trying to find ways and means of saving the life of Chinnaswami through that medium. She approached a renowned astrologer, Narayanswami Iyer, gave details of Ramanujan's horoscope (from memory) and wanted to know how long he was going to live, but she did not mention her son's name. The astrologer, after some study, told her that according to the horoscope, the person would either die young at the height of his reputation or live long in obscurity. But, "who is the person?" he asked. When she said that the person was Ramanujan, Narayaswami apologised for breaking the astrologers' code of conduct never to give shocking information to the questioner straightaway and he requested her not to tell Ramanujan's relatives. At this Komalatammal burst into tears and told the astrologer that she herself was the unfortunate mother of Ramanujan. Then the astrologer wanted to study Janaki's horoscope in order to find out if there could be any mitigating factor. Komalatammal gave details of her daughter-in-law's horoscope orally just as she had given correct description of her son's horoscope. After half an hour of

frantic search Narayaswami Iyer exclaimed "there is only one mitigating influence. It is desirable that Ramanujan does not live with his wife for a few months". Komalatammal, who herself was an astrologer, said "I have seen it myself. I have been pleading with my son to send Janaki to her parents home for some months. But Chinna who has always been obedient, does not listen". In any case this astrological prophesy may or may not have to do with the time of Ramanujan's passing away, but it definitely aggravated the not-too-good mother-in-law-daughter-in-law relation. For that matter an astrological prediction about Ramanujan's short life was made earlier also. M. Govindaraja Iyengar, who had taken tuition from Ramanujan says that his father who was a well known astrologer of a rationalistic school, on examining the horoscope found that Ramanujan would be a renowned scholar in English during his thirty-second year and since that was impossible it might indicate that his life might end in that year". It would be difficult to say anything about the firmness or otherwise of the axioms on which these computations were based, but the effect on the minds of concerned persons is obvious. If by any chance, Ramanujan had heard from Govindaraja about the prediction, there is every likelihood that on that basis during his illness he kept on telling people about his death in the thirty third year.

Realising well that his days were numbered he was spending almost his entire time on his so-called mock-theta functions. One week before moving from Kumbhakonam to Madras, he wrote to Hardy

"I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I shall call "Mock Theta Functions". Unlike the "False Theta Functions" (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending with this letter some examples...

Mock theta-functions

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \dots$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \dots$$

(5th order)

$$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \frac{q^9}{(1+q)(1+q^2)(1+q^3)} + \ldots$$

(7th order)

$$\text{(i)}\ 1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)} + \ldots$$

The untiring soul of Ramanujan mocking at the body that had melted away kept on working on mock-theta functions till less than 90 hours before the last moment came on the 26th day of April 1920 at 10.30 a.m.

As soon as the candle of life was extinguished, Namperumal went to R. Ramachandra Rao's bungalow and requested Mrs. Rao and her sons-in-law to arrange for the funeral. At 1 P.M. the same day Ramanujan was cremated according to strict religious rites. There was not a big crowd : five brahmins, Ramachandra Rao's son-in-law (Rama Rao), an advocate C. Rajagopalchariar (also a brahmin), Namperumal Chetty, Komalatammal, Janaki and his brother Lakshimanrasimhan watched the body of Ramanujan being consigned to flames. Dr. Chandrasekhar noted in his diary as quoted by Rankin in his paper entitled 'Ramanujan as a patient' that it was a tragedy "a tragedy which is too deep for tears. I cannot get over a particular experience–in fact it was an unusual experience for a physician. Tragically enough Ramanujan himself told me that he had lost the will to live on several occasions. Again Ramanujan told me he should not have returned to India".

When the news reached Cambridge, many were taken aback. Neville, with whom Ramanujan stayed for some week on arrival at Cambridge noted "Had Ramanujan not left India he might be alive today; but he would have had always the sense of weakness and frustration, not of power and accomplishment. Death too, was a frustration, but the life's purpose of which his mother dreamed was at last in part fulfilled and it is *better to be frustrated by unsought death than by life*...Srinivasa Ramanujan believed for he told me, just before he left England, that he had never doubted

that he did well to come". Hardy also wrote to Registrar Dewsbury on 26th May "It was a great shock and surprise to me to hear of Mr. Ramanujan's death. When he left England the general opinion was that, while still very ill, he had turned the corner towards recovery, he had even gained over a stone in weight (at one period he had wasted away almost to nothing). And the last letter I had from him (about 2 months ago) was quite cheerful and full of mathematics...There will of course be universal regret among European mathematicians at such a tragedy-his work was *just beginning to be known abroad, especially in Germnay.* Prof Landau of Gottingen, Porf, Cohn of Berlin were both interested, and impressed by it".

Just at a moment when the broader mathematical community, began to take interest in his work Ramanujan left the world. One is reminded of a couplet by an Urdu poet

Bade shauq se sun rahā thā zamānā
hamīn so gaye dāstan kahte kahte

(the world was listening with rapt attations, I myself fell asleep telling the story).

Vitwāngal Chinna,
(Farewell ; Chinna),
Vanakkam Ramanujan ;
(Salutations ; Ramanujan)

12

My Dead Bullock Had Big Eyes

There is a story that a foolish farmer had a bullock. He used to ride it to town with heavy load but would give very little fodder. All that he used to give it was lashes after lashes. One day the emaciated bullock fell down on way to the market, never to rise again. Some passer by remarked "What a pity, the poor thing is dead". The foolish farmer started crying "My dead bullock had big eyes" and could not stop sobbing.

Ramanujan was born in poverty, worked in poverty, struggled with empty stomach and when he got recognition and proper means of livelihood abroad, it was too late and then recognition and honours began to rain in his own country on a body that was no more in this world. Remembering Ramanujan, Narayana Iyer's son N. Subbanarayanan lamented. "It is really unfortunate that it was left to foreigners to recognise the real geniuses of our land and *present them back to us* with great honours which they could not have obtained here".

It would be wrong to say that Madras University closed its doors on Ramanujan for ever because an affilliated college had found him "unworthy", after all Sir Francis Spring with the help of his brother-officers was able to secure a place for him as the first research scholar. Littlehailes, after becoming the Director of Public Instruction moved the University to establish a professiorial Chair for him, when Ramanujan had got the

Fellowships of Royal Society and Trinity College and Hardy had asked the University to make some permanent arrangement for him. Ramanujan could not occupy that chair. But his oil portrait was unveiled by the Vice Chancellor Sir. K. Srinivasa Iyengar in 1922 fixed in the Syndicate Room. In 1930, members of the Indian Mathematical Society collected a sum of Rs.18,000/- for instituting Ramanujan Prize to be administered by the University of Madras. The first recipients of this Prize were D D. Kosambi and S. Chandrashekhara, whose papers on Differential Geometry and Astronomy respectively were equally good. In the subsequent years the Roll of Honours included such names as S. Sivasankaranarayana Pillai, B. Ramamurthi, Venkatarayudu, Meenakshisundaram, B.S. Madhava Rao, V.S. Krishnan and B.R. Srinivasan. It appears that with the exception of Kosambi, on one from any region in India other than the Southern aspired or tried for this prestigeous prize although by mid-fifties the number of Universities and research scholars in mathematics had swelled all over the country.

The most important tribute to the memory of Ramanujan was paid by a businessman Alagappa Chettiar who had opened a College of Technology at Karaikudi, town about 500 kilometers to the south-west of Madras. At the same place was born K.S. Krishnan, one-time Professor of Physics at the University of Allahabad and later Director of National Physical Laboratory. He induced Alagappa to establish an institute of advanced studies named after Ramanujan and promised to find a suitable person to be the Foundation Director for it. When he held promise about finding a Director, he had a close, friend of his, T. Vijayaraghavan in mind. Vijayaraghavan and Krishnan were classmates at Presidency College Madras, but the former had heard the canard somewhere that Ramanujan had failed in Mathematics, so the idea entered his head that by neglecting other subjects and concentrating on Functions of Complex Variable he could become another Ramanujan. The result was that he failed in his Honours examination. But he did engage himself in Research; he had no financial problem. By sheer luck his teacher, Ranganathan, happened to be in Oxford during March 1925 where Hardy was Savilian Professor of Geometry at New College. Ranganathan

showed a paper on Tauberian Theorems by Vijayaraghavan to Hardy. Hardy was impressed and exclaimed "a remarkable mathematician from your country, next to Ramanujan". Hardy wanted to have him but the Secretary of Delegacy, who was a retired officer of the Indian Civil Service, raised a technical objection about eligibility of the candidate as had failed in B. A. Honours. But Hardy managed to overrule the objection and Vijayaraghavan was able to join New College. On return, he worked in Undivided India and at the time of Partition in 1947, he was working at the University of Dhaka (in present day Bangla Desh). After Partition he moved to Andhra University, Waltair, but his relations with the Vice Chancellor were not very happy.

Krishnan managed to bring Vijayaraghavan to Madras and on January 26, 1950, the day India declared herself to be a Republic the Ramanujan Institute was founded and housed in a private building-separate and unconnected with Madras University. For quite some time the Institute was a one-man affair and attempts were made to have at least one more eminent mathematician. Ranganathan suggested the name of a brilliant mathematician S. Sivasankaranarayana Pillai, who had shared with B. Ramamurti the Ramanujan prize in 1937 for his paper "Warring's Problems with Polynomial Summands" for the job. Pillai at that time was a Lecturer at Calcutta University and was invited at the Institute of Advanced Studies, Princeton (U.S.A.). But he perished in plane crash near Qahera (Cairo) and thus Ramanujan Institute was deprived of the services of a brilliant Number Theoerist who had done a good deal of work on the chapters in Ramanujan's Notebook connected with that area.

For quite some time the Institute was functioning with one Professor, one Lecturer and two Research Scholars. Later C.T. Rajagopal, who was working at Madras Christian College joined also. But the Institute was not getting any financial aid, so Alagappa Chettiar wanted to shift it to Karaikudi. In 1955 Vijayaraghavan died and Rajagopal became the Director. Chettiar also died in 1957 and, in order to overcome the financial difficulties, Rajagopal approached the Finance Ministry, Govern-

ment of India. The same year the Institute was taken over by the University of Madras, but it retained its seperate existence and the Ministry of Education provided finances. In 1967 the University Grants Commission of India proposed to convert the Institute into an institute of advanced studies in mathematics. The University readily accepted the proposal and the University Department of Mathematics was "coalesced" with the Institute to form the Ramanujan Institute for Advanced Study in Mathematics. Since then money has been coming from various funding agencies, especially the University Grants Commission. It has grown over the years; a number of seminars, Workshops, Special Lecturers and visits from eminent mathematicians are held regularly. The Institute has built up contacts with many important centres of advanced research both in India and abroad. Research in various branches of modern pure mathematics, is undertaken but if the Institute has to justify use of Ramanujan's name attached to it, it will have to devote more attention than at present to the Works of Ramanujan and not leave it alone to mathematician like George Andrews, Bruce Berndt, Hirschhorn, Richard Askey abroad and a few in India.

Apart from providing some financial help to the Institute, the National Institute of Sciences, India (re-christened as Indian National Science Academy) also instituted a Ramanujan Medal for original research. The roll of Honour includes, S. Chandrasekhar, B.P. Pal, K. Chandrasekharan, P.C. Mahalnobis, G.N. Ramachandran and Harish Chandra.

Another recognition of Ramanujan's memory came in the form of financial help to Janakiammal. It may be recalled here that Ramanujan had requested Madras University to spend his money, after deducting the expenses on his family and himself, for the benefit of poor students and orphans. But no records are available to show if the University did anything in this connection-only this much is certain that no scholarships, either at the University or schools were awarded from Ramanujan's dues. As Ramanujan's death occured on the 26th of April, his stipend for twenty six days was due, one of his brothers claimed that money from the University, but Seshu Iyer intervened on behalf

of Janakiammal and got the money for her. Ramachandra Rao also gave her Rs. 20/– for a few months. The University itself started giving her Rs. 25/–. Then the amount was increased to Rs. 75/– then Rs. 150/–per months and now stands at Rs. 300/–. Seshu Iyer also arranged to get Rs. 1,000/- paid to her from an amount of Rs. 5,000/– due to Ramanujan from Britain. Indian National Science Academy sanctioned Rs. 1,000/- per month. Some other governmental bodies also provide her financial support. But she says that *only the India National Science Academy payments are regular and are received punctually* on the Ist of every month. Others take, sometimes 5 to 6 months. She recalls that there was a time after her husband's death that no one cared for her. It was thought that her parents were well-off. so they would look after her. But she was pulling on by stitching clothes and living in a room connected with a temple. She was unhappy then as she was poor and now that her monthly income is about 2 to 3 thousand, she is more unhappy as some relatives, thinking her to be a rich lady living alone, pester her for money.

The Posts and Telegraph Department of India paid tributes to the memory of Ramanujan by issuing a postage stamp with Ramanujan's picture on it. The occassion was the 75th birth anniversary on 22nd December, 1962. On the same day talks about his life were broadcast from various stations of Akashvāṇī (All-India Radio). Later a ship was also named after Ramanujan, a photo of which is hanging on a wall of the house (45, Muthia Mudali II Street, Royapettah, Madras, 14) where Janakiammal lives presently. In the house there is beautiful statue of Ramanujan which Richard Askey presented a few years ago. Janaki says that when she looks at the statue it appears to her that Ramanujan has come alive.

But the Corporation of Madras seems to have never heard of Ramanujan. Streets are named after various religious leaders, and politicians of different shades. New colonies have been named after the Prime Ministers of India. The scientist Chandrasekhara Venkata Raman also has been honoured by giving his name to a street. But not Ramanujan—there is no

Ramanujan puram, no Ramanujan Salai, no Ramanujan Street, no Ramanujan Lane, none. But Kumbhakonam still remembers him. A hall of Kumbhakonam Town High School is now Ramanujan Hall and on a request of the present Headmaster Sri R. Viswanathan, Kumbhakonam Municipality has passed a resolution that "Hereafter the Town High School Road will be named Ramanujan Town high School Road".

coffee, Ramanujan asked him "if four people sit on each side of the table, how many can you serve at the same time ?" The waiter said "simple, sir, sixteen". Ramanujan smiled and added "No, only twelve, you will be standing on one side with no one else sitting there." He retained this style of humour in later years also. As Chintaman Deshmukh, who was a contemporary of Ramanujan at Cambridge, says "Occasionally I used to have a Sunday meal with him at which he apologised in a solicitious way about the spicy *rasam* (a liquid concoction made of tomato and some other vegetables) that he had prepared for his friends ... The meals so served were quite delicious, especially as Ramanujan used to regale us with mathematical conundrums and puzzles which he took care to ensure were not above the heads of his non-mathematical friends." E.H. Neville also had the same experience. According to him Ramanujan "had a fund of stories, and such was his enjoyment in telling them that in his great days his irrepressible laughter often swallowed the climax of his narrative. He loved a paradox, but I do not know of any that he invented; the paradoxical element in some of his early work must have made him aware that the wildest nonsense of today may receive a logical interpretion tomorrow."

So strong was his habit of laughing off his worries that even when there was no hope of his survival, he gave a new meaning *Than-sa-voor* (place ofmone's death) to Tanjavoor and refused to go there. Also even in his last days he used to console his wife by telling her all sorts of funny, real or imaginary, anecdotes about his follies in cooking while at Cambridge.

Besides his sense of humour influencing his life was his firm belief in the doctrines and rituals of the religion in which he was born. Peripheral concomitants like prophetic dreams, seeing great souls in the ravings of a mad man, palmistry and astrology, all, have been claimed to be the ultra-natural powers of Ramanujan by his friends but there has been no unanimity. Anantharaman says "It is not correct to say that he knew much of astrology though his mother knew something of it. But Ramanujan had some intuition in interpreting dreams." A college-mate N. Hari Rao, says that the only subject in which he

13

Puffed Mouth ? Never

It is generally believed that the possibility of a smile appearing on the face of a pure mathematician is less than epsilon. But only one counter-example is sufficient to explode this myth. Ramanujan was interested in so many things besides mathematics, his aversion for Physiology and Greeco-Roman History notwithstanding. As K. Anand Rao puts it "he was a man full of humour and a good conversationalist and it was a pleasure to hear him." In fact it was his sense of humour that gave him strength to face hardships smiling. It never left him. In childhood he indulged in pranks like taking a Saivite boy to the Vaishnavite temple by making him look like a Srivaishnavite with *namam and sricharanam* on the forehead and laughing all the way back home when the mischief was detected. There are numerous anecdotes of this type at every stage of his life. When any inconvenient thing happened, he would simply laugh it away. In his youth, when he was a clerk at the Port trust, once, he was engaged in discussion with a friend late at night. Another person, not interested in the talk felt disturbed in his sleep, took a pitcher full of water and poured the entire contents on Ramanujan's head saying "take it, this will cool your heated brain." An unperturbed Ramanujan exclaimed "one more, please, you are giving me *gangāsnānam* (a dip in the holy waters of Ganga).

Once he was taking coffee in a small shop with some friends who were sitting round a square table. When the waiter brought

took any interest besides mathematics was palmistry. But a few more specific instances have been recorded. For example, T.K. Rajagopalan, a high official of the Board of Revenue, Madras Presidency, has noted (as recorded by Ranganathan) that in a room of his servants' quarters where a *sanyāsin* (monk) was staying, Ramanujan spent the night as he was unemployed and had no place to go. In the morning a happy Ramanujan came to tell his host about the *sakthi* (spiritual power) of the *sanyasin*. It was due to his presence that "while asleep I had an unusual experience. There was a red screen formed by flowing blood as it were. I was observing it. Suddenly a hand began to write on the screen. I became all attention. The hand wrote a number of results in Elliptic Integrals. They stuck to my mind, as soon as I woke up, I committed them to writing." As an explanation for such a phenomenon, Rajagopalan asserts "Ramanujan was an ardent devotee of Narasimha (the lion faced *avatar* of God) the sign of whose grace consisted in drops of blood seen during dreams. Ramanujan stated that after seeing such drops, scrolls containing the most complicated mathematics used to unfold before him and that after waking he could set down on paper only a fraction of what was shown to him "

A much more striking phenomenon about Ramanujan demonstrated fourteen years after death has been recorded by Ranganathan in his *Ramanujan the man and the mathematician* (p. 16) :

"It was 1934. I was with K.S. Krishnaswami Iyengar, a leading advocate of Madras and later a judge of the High Court. He had known Ramanujan as a boy at Kumbhakonam. With the help of the Ouja Board he invoked Ramanujan. There was response. I asked him whether he was continuing his mathematics. He replied 'No, all interest in mathematics dropped after crossing over.' I asked him what he was occupying himself with. The reply was 'Meditation. Studying *Vishnu-sahasranāmam* (a hymn of 1008 names of God). I then told him about his third Notebook which was in my possession in the University Library. Only a few pages of it had been filled up. It was all a series of tabulated numbers. Nobody could make out what it was all

about. Could you throw some light on it ? Ramanujan replied 'I do not remember about it. Bring the Notebook to this place this time next week. I shall look into it. If I recollect anything, I will tell you'. The Notebook was taken to Krishnaswami Iyengar's house at the appointed time. Ramanujan was invoked. He asked me to turn the pages slowly. Then came the reply 'I now remember. I was working on Mock-theta functions."

It would be unfair to dismiss these narratives as the figment of Ramanujan's *bhaktas* (devotees) imagination, but to expect universal acceptance would be equally unfair. Besides there is a diametrically opposite view of Ramanujan's life presented by Ashis Nandy in an article entitled 'Ramanujan in England. An East-West Collaboration in Science'; *The Psychoanalytic Review* 66 (1979) 423-442. As students of Mathematics, generally do not have access to this journal copious excerpts from the article are given, here. Nandy asserts :

"The most remarkable aspect of Ramanujan's encounter with the West was, however, the relationship he established with Hardy ... This raises a number of questions. For instance, were the bonds between Ramanujan and Hardy *solely intellectual and scientific ...?...*

Hardy had strong homosexual tendencies.... Ramanujan on his part, with his ambivalent feelings toward an interfering and possessive maternal authority and his emotional distance from his father, found in Hardy for the first time an interevening nurturant and close male authority who did not trigger his bivalence toward all combinations of intervention, nurture and proximity ... The man of religion in India was to be little more bisexual...."

The present writer being an ordinary mathematical worker has not been trained in the use of psychoanalytic tools, so it will not be proper to engage into polemics with Nandy. However let Rankin speak for Hardy. Rankin was one of Hardy's research students, and as he mentioned in his article entitled "Ramanujan as a Patient" :

It goes without saying that Rankin's observations about Hardy are *based upon personal experience of long standing*, while Nandy's pronouncement relies mostly upon private conservation with people whose opinions about Hardy may or may not be objective. Besides, he often draws conclusion from some contemporary writings, but an example will show how *purposefully* such writings are utilised: It may be noted that Nandy asserts "One of the areas of psychological functioning in which Hardy manifested his femininity was his intense, uncompromising pacifism (p.433) and in Notes adds "*Hardy did not join the war effort and stayed back* in Cambridge *because of his pacifism*". (p.447). It is really interesting to note that Nandy, who is so fond of giving copious quotations from C.P. Snow chooses to skip over the statement about Hardy in the preface "*he could not find satifactory basis for conscientious objection*; his intellectual rigour was too strong for that. In fact he *volunteered* for service under Derby Scheme and was rejected on medical grounds". Obviously there remains no need to comment. I must admit that I have very reluctantly included Nandy's article because it paints an unworthy picture of Ramanujan when the whole world is honouring his scientific contribution.

It would have been much better if Nandy utilised his expertise in psycho-analysis on examining the claims by some admirers of Ramanujan about his *transrational powers* (manifested during his life time and also posthumously) instead of toiling to "prove" a saucy hypothesis by marshalling all sorts of arguments which may or may not have any relevance in the case of Ramanujan. In any case the *dictum* that a boy whose mother is domineering begins to imagine that he *himself is a mother*, may be a self-evident truth for some particular sect of psychoanalysts, but can not be applied indiscriminately without gathering concrete evidence. Going through the lengthy analysis in Nandy's paper one wonders if here is a serious researcher who has studied the holy books of various sects belonging to Hinduism, who knows the difference between Tantrism and *Sakhi-sampradaya* on the one hand and the doctrines of various schools of Vedantism on the other. Or, has he any idea about the beliefs and rituals of the Saivites and Srivaishnavites (to which sect Ramanujan

belonged). Or, if Nandy is a modernist, has he based his *verdict on bisexuality of religious Hindus* on any recent anthropological, sociological medical or psychological survey recently *conducted in India* on a vast scale or on his own personal knowledge of a sizeable number of such people. Or, is he out to find some names of V.I.P.'s as *specimens* for the hypothesis contained in Freudian gospels and the peripheral literature published in the United States. He wanted to catch Littlewood also encouraged by Hardy's remark "I owe more to him (Ramanujan) than anyone else in the world *with one exception*, and my association with him is the one romantic incident of my life". But Nandy exonerated him as "Littlewood was a more hardy, overly masculine specimen" (p.441). After all Littlewood had joined the war as a Second Lieutenant and was immune to barrack-syndrome. In any case it sounds like a bold exercise in 'character assassination' by some *avatar* of Miss Mayo (whose *Mother India* was characterised by Gandhiji as a drain-inspector's report) speaking again after half a century.

There is however, one aspect of Ramanujan's life where both the supporters of transrationalism and Nandy do not seem to have any divergence-and that is his interpretation of mathematical entities '*zero*' and *infinity*' which according to the former is *realisation of Indian spirituality* and according to the latter *ritual neutralisation of Western Science*. The so-called Theory of Reality that, according to Mahalnobis Ramanujan based on interpreting *zero* as a symbol of *Nirguna Brahman* (Absolute attribute-free Existence) and infinity as the totality of all possibilities, which was capable of becoming in reality and which was inexhaustable-or in the words of Ranganathan *Sarvguna Asray* (Abode of all attributes). The product of infinity and zero would supply the whole set of finite numbers. Each act of creation...could be symbolished as a particular product of infinity and zero, and from each such product would emerge a particular individual of which the appropriate symbol was a particular finite number".

There is nothing unusual or surprising in Ramanujan's attempt at propounding a metaphysical theory using mathematical

terminology. Rene Descartes, who based his metaphysics on the dictum *esse est percepi* (the essence of existence is perception) believed in some Existence that percieves the entire Creation at the same time. There was a dualism of Mind and Matter and just as in Euclidean Geometry parallel straight lines meet at 'infinity' Mind and Matter also running parallelly meet at infinity. Leibniz, believed that all that one needs to know is *zero* and *one*. He asserted that God is 'one' and He created the Universe *ex-nihilo* (out of nothing) which is 'zero'. When he announced his 'discovery', a medal was struck to commemorate this achievement. And today the computer has adopted this principle for representing numbers. In India also some mathematician applied this technique but in the reverse direction - using metaphysics to prove a mathematical theorem. In Sanskrit a synonym for zero, was *kha* and for infinity *kha-cheda* or *kha-hara* (zero-denominator). In order to prove that $\infty \pm n = \infty$ where n is a finite number, Bhaskara II (1114—1200), argues "In this quantity consisting of that which has cipher for its division, there is no alteration, though many may be inserted or extracted, as no change takes place in the infinite God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth".

A possible explanation of Ramanujan's exercise in Metaphysics could be that at some stage he also believed that infinity was any finite number divided by zero, hence $0 \times \infty = 0 \times (n/0) = n$. This seemed to be a continuation of what he said at primary school that $\frac{0}{0}$ = any number. However, he does not seem to have elaborated or developed his metaphysics further. On the other hand he told Hardy that his religion was a matter of observance and not of intellectual conviction and all religions seemed to him more or less equally true. Also, on return to India he told a childhood friend that the gods that he worshipped earlier did not exist. All this indicated that over the years he was changing, but it would be difficult to say if he had really changed. The process of change had possibly begun. And there is nothing surprising in it Change seems to be the Law of Nature. As a historian of science, George Sarton, puts it "As long as people stagnate in their ancestral

villages they do not ask themselves many questions, for every query has been asked and answered a number of times and it is no use worrying about it any longer. On the contrary, when people of different races and with different traditions come together, it must occur sooner or later to the most intelligent of them that there is more than one way of looking at things and of solving problems. If they are intelligent enough they must wonder whether their own traditional solutions are the right ones, or they may realise that things which they had never thought of questioning are questionable".

This law of change did apply to Ramanujan also. He was neither a divine being nor a filthy character (as Nandy would us believe). He was a human being with all his weaknesses and strength. In any case he was a hard-working, creative, self-made Indian who forced his way into the fold of the great. What he accomplished may not be too much in some eyes, but the unfinished jobs that he has left are mindboggling for quite a few generations to come. With his death all his personal faults are also dead, only drain inspectors' may be interested in raking them up. He will be remembered for what was good in him, as he had harmed nobody. As Neville remembers him "He had serious interests outside mathematics, and was ready to discuss politics or philosophy....In every day life, he had an instinctive perfection of manners that made him a delightful guest or companion. Success and fame left his natural simplicity untouched. To his friends he was devoted beyond measure, and he devised curiously personal ways of showing his gratitude and expressing his affection. The wonderful mathematician was indeed a lovable man". To this picture Hardy adds the picture "of a man who had his peculariries like other distinguished men, but a man in whose society one could take pleasure, with whom one could drink tea and discuss politics; the picture in short was not of a wonder from the East, or an inspired idiot, or a psychological freak, but of a rational human being who happened to be a great mathematician".

14

Treasure in Scribblings

About two thousand years ago in India mathematical computations were performed on *pāṭī* (board) with a piece of chalk or spread on *dhūli* (dust or sand = *ghobār* in Arabic–which gave the name ghobar numerals' to Indian numerals in Europe) and *pāṭī-ganita* to Algebra in Sanskrit. Mathematicians would then express their theorems in verse and note them down on palm-leaves the idea being that expression was only an aid to the mathematician's memory, generally proofs were either broadly indicated or skipped. This tradition of using a *pāṭī* for computation continued for many centuries. By Ramanujan's time the *pāṭī* was replaced by the slate and as Boyer says "Srinivasa Ramanujan had the same uncanny manipulative ability in Arithmetic and Algebra that is found Bhaskara", he had no difficulty in working out lengthy computations on a slate. He would then note down his results on any paper that he could lay his hands upon-scrap paper, wrapping paper, cheap paper sometimes he would fair out these sheets arranging them in chapters whereever possible and would get them bound in the form of a Notebook. He developed this habit while still at school and continued it even on his death bed. It has not yet been ascertained how many Notebooks he left, but Sir Gilbert Walker is said to have seen 4–5 such Notebooks.

When Ramanujan died, a box, in which some of his papers and also jewels were kept, was according to Janakiammal missing. However, whatever papers were left were acquired by Madras

University. As compensation she was given a monthly pension of Rs. 20/-which after some years was increased. These papers contained two bound Notebooks also, one complete and the other only a few pages. At least one Notebook was left at Cambridge itself when Ramanujan returned home.In 1924, Ranganathan, the Madras University librarian, happened to be in Britain. He met Hardy who was then at New College, Oxford. Talking about the papers of Ramanujan, it turned out that a Notebook was left with Hardy. Ranganathan wanted to know more about it, but Hardy took out the Notebook and handed it over to the former saying "Ramanujan belonged to your country. This treasure should also be in your country". Ranganathan contacted Madras University for instructions. The reply was simple : "Keep the Notebook with you". The British Museum library offered £500/- for it but Ranganathan refused to part.

Hardy asked Ranganathan to persuade Madras University for getting the Notebook published after some editing. He himself confessed that the job was too heavy for him, "Actually I experimented with working on one chapter of this Notebook-the chapter on Hypergeometric Series (which was published in *Proceedings' Cambridge Philosophical Society* 21 (1923) p. 492-503). This work has taken me several weeks. If I were to edit the entire Notebook it will take the whole of my life-time." On return to India Ranganathan deposited the Notebook that he had brought from London by the side of the other two Notebooks that were procured from Janakiammal. On comparison it was found that between the thicker Notebook and the London Notebook there was much in common but still they are not the same. Ranganathan enquired from Ramachandra Rao about it who said "I asked Ramanujan to make a copy of the Notebook for the sake of safety, but it is so difficult for a creative thinker like Ramanujan to make a faithful copy as clerks of the sub-registrar's office are expected to do. Evidently he went on improving the chaoters while copying them. Finally he began to enter altogether new results in new chapters". Finally three Notebooks were housed in Madras University Library. Notebook 1 contains 16 chapters covering 134 pages out of which 80 pages are devoted to an assortment of various topics. Some chapters are arranged in such a way that often the topics discussed therin are disjoint. Notebook 2, which is a revised and

enlarged version of Notebook 1 consists of 252 pages covering 21 chapters, and Notebook 3 gives miscellaneous results in 33 pages.

The University of Madras could find neither a 'competent mathematician' nor funds to edit and publish the Notebbooks. Possibly there were no people like Sir Francis Spring and Lord Pentland in the University or Government to take up the matter rigorously. The University however provided some funds to the Librarian for making three copies each of the three Notebooks. On completion one set was sent to Hardy. Hardy agreed to publish the *Collected Works of Ramanujan* but due to paucity of funds was satisfied with publishing the *Collected papers* only, giving copies of Ramanujan's papers publisned in India and abroad besides the Questions and Solutions that appeared in the *Journal of Indian Mathematical Society*. He also opined that two-third of Ramanujan's work done in India was 're-discovery'. But Bruce Berndt after his extensive studi s does not agree with this estimate. He is of the view that "Some original results in the Notebooks have since been re-discovered by others usually without the knowledge that their theorems are *found in the Notebooks*"

According to a footnote of a paper entitled 'Two assertions made by Ramanujan; *Journal London Mathematical Society* 3, (1928) p.233; G.K. Stanley indicates that by then Hardy had abandoned his earlier intention of preparing Ramanujan's material for publication, to which Rankin in his paper 'Ramanujan's Manuscripts and Notebooks ((*Bulletin. London Mathematical Society*, 14 (1982) p.81–91) added "but he (Hardy) did of course, publish several papers based on different aspects of Ramanujan's work at different times...However by 1928 G.N. Watson who had gone to Birmingham in 1918 as Mason Professor of Pure Mathematics, had entered his Ramanujan period between 1928 and 1931, he and his colleague C.T Preece published fourteen papers under the general title. 'Theorems stated by Ramanujan'. In all nearly thirty of Watson's papers owe *some inspiration to the work of Ramanujan*. It is no doubt that later in 1928, Hardy passed on to Watson his Ramanujan material". After the Notebook 2 was passed on to Watson he decided in 1928 to work together with B.M. Wilson of Liverpool University, for editing it in a publish-

able form; chapters 2 to 13 were to be taken up by Wilson and 14 to 21 by Watson both working independently. Wilson died young at the age of 38 in 1935. In less than six years he was able to prove most of the formulas in chapters 2–5, one–third of the results from chapter 8 and a half from chapter 12, but he could not publish anything. Waston, on the other hand concentrated on chapters 16-21 but only published the papers that were inspired by them. However his interest in the Notebook 'lapsed' by 1939.

When Watson died in 1965, Rankin visited the former's house in order to write an obituary for London Mathematical Society, of which Watson had been President from 1933 to 1935. There Mrs. Wastson showed him "a great deal of material relating to Ramanujan, including copies of Notebooks 1 and 2, his (Watson's) work with B.M. Wilson on the Notebooks and much other materials" Later, J.M. Whittaker 'unearthed' a second batch of Ramanujan material when he visited Mrs. Watson in November 1965. The entire material was donated to Trinity College Library, which has been properly catalogued. Only recently, it has been found that a copy of Notebook 2 in Watson's handwriting as well as some copies of other Ramanuian manuscripts are available in the Mathematical Institute of Oxford also.

However, back home no attention was being paid towards publishing the Notebooks. Only Ranganathan was trying without success. After Independence when Sir Maurice Gwyer, who had earlier cancelled the Degrees of Dhruva Sanghi and Muqimuddin Farooqi for participation in Freedom movement, was the Vice Chancellor of Delhi University, Ranganathan approached him also. Sir Maurice remarked" it is a shame that the Notenook has been left unpublised over twenty years. If you get the Notebook, I shall get it published. I will send my hat round if need be". In 1948 at the Commonwealth Universities Congress at Oxford, Ranganathan raised this problem with Dr. Radhakrishnan, Dr. Wali Mohammad and some other Indians, who agreed in principle but could not work out any formula for arranging the necessary funds. Finally in 1949 Madras University was able to provide funds only for making three photostat copies of the three Notebooks. A tentative list of editors was prepared and relevant pages

were to be sent to them for study. One complete copy was given to Sivasankarnarayana Pillai, who was going to the Institute of Advanced Studies in Princeton, but due to his death in a plane crash (1950) the project itself was dead. Ranganathan then moved a resolution in 1954 at the Delhi Session of Indian Mathematical Society for taking up the job of publishing the Notebooks. After discussion and approval of the resolution, it was decided to seek help from Tata Trust. Accordingly, Tata Institute of Fundamental Research *took the first step in this direction.* In 1957 the Institute brought out 1,000 photostat copies of the Notebooks in two bound volumes-the 1st volume consisted of the 1st Notebook and the 2nd contained Notebooks 2 and 3. There was no editing, however, it was a great service to the cause of Mathematics that at least the original text of Ramanujan maoterial could be within reach of the wider community of mathematicians.

Besides the above three well-known Notebooks, another notebook has been unearthed. In 1976, Prof. George Andrews of Pennsylvania (U.S.A.) was on a visit to Cambridge. To his delight he found a 140 page manuscript in Ramanujan's handwriting at the Trinity College Library kept ina a box that Mrs. Watson had given. On examination, it was found to contain more than 600 formulas including all the results on mock-theta functions that Ramanujan had communicated to Hardy in his last letter. Andrews infered that the document was prepared in the last days, but instead of treating it as the "*Last Note book* called it the *Lost Notebook of Ramanujan.* The results of his early investigation appeared as "An Introduction to Ramanujan's Lost Notebook' in *American Mathematical Monthly* 86(1979), 89–108. In October 1981 he presented an elaborate account of the contents at the 51st Session of National Academy of Sciences (India) held at Cochin University. Andrews is of the view that some results from this Notebook suggest combinatorial aspects of modular equations that were previously unsuspected. Detailed proofs of all the results are not always given in the original and in some cases proofs have been provided by George Andrews. For example :

Let C(q) denote the continued fraction

$$1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{+ \cdots}}}$$

then the identities

$$\text{(a)} \qquad \frac{1}{(C(q))^3} = \frac{\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{5n+3}}}{\sum_{n=0}^{\infty} \frac{q^{n}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} \frac{q^{4n+3}}{1-q^{5n+4}}}$$

$$\text{(b)} \qquad \sum_{n=0}^{\infty} \frac{q^n}{(1+q)(1+q^3)\ldots(1+q^{2n+1})}$$

$$= \sum_{n=0}^{\infty} (-1)^n \, q^{6n^2+4n}(1+q^{4n+2})$$

have been fully proved in the *American Mathematical Monthly paper* mentioned above.

Recently Professor Bruce C. Berndt of Urbana, Illinois has embarked upon a mission to publish the Notebooks with detalied comments and proofs where necessary. For this purpose he made a 'pilgrimege' to India, paid his respects to Srimati Janakiammal and gathered whatever information possible about Ramanujan's works. First he published a number of articles in many journals, including *Journal of Indian Mathematical Society* devoted to detailed analysis of different Chapters from the Notebooks. He also, published the hitherto unpublished '*Quarterly Reports* submitted by Ramanujan to Madras University when he was a research scholar. Berndt has planned to bring out three volumes dealing with the

three Notebooks. The first volume has already appeared in 1984 (Springer). In the preface he writes "Many of the results found herein are very well-known, but many are new. Some results, are rather easy to prove, but others are proved only with great difficulty. A glance at the contents indicates a wide diversity of topics examined by Ramanujan. . .We hope that this volume and succeeding volumes, will further enhance the reputation of Srinivasa Ramanujan, one of truly great figures in the history of mathematics In particular Ramanujan's notebooks contain new interesting and profound theorems that deserve attention of the mathematical public". And, as R.A. Askey points out "more needs to be done than Berndt has done, for some of these results look like they will be very useful, and Berndt has in general not looked for applications. However, he has done so much more than anyone else in helping us start to understand the work in these notebooks.

This task, however, has not yet been taken in right earnest in India. A stirring appeal was made by Professor K. Venkatachaliengar in his Presidential address at Ranchi Session of Indian Mathematical Society in December 1979. He spoke with emphasis :

"The task of *getting the originals*, or at least photo copies, of all the Ramanujan manuscripts must be undertaken *by our countrymen* who should also face the formlidable task of editing them, taking first those parts that are comparatively easy to edit; this is true of his elliptic function formulae which Ramanujan has given in nearly half of his notebooks. May I hope that the University Grants Commission will engage young scholars to perform this task as early as possible" He again repeated his appeal at the next session (46th) at Bangalore (1980) making it more specific. Concluding his Presidential address he said "We earnestly solicit the help of the Central and State Governments and other philanthropic organisations to help the Society (Indian Mathematical Society) develop a Ramanujan manuscript publication fund. . .The interest from this amount can be utilised towards the preparation of photostat copies of the remaining manuscripts in order to make them available at moderate cost to

interested scholars". This call was given out a decade ago and in the meantime many welcome developments have taken place in the country—the National Board for Higher Mathematics has been set up, more funding agencies are now prepared to encourage advancement of mathematics, more mathematicians have access to centres of Advanced Studies abroad. Let us hope that steps will now be taken to bring out *Collected Works of Ramanujan* one day.

15

When the Sun Goes Down

India and the world remembers Ramanujan, but in what way ?

Jawahar lal Nehru once observed "Mathematics in India inevitably makes one think of one extra-ordinary figure of recent times. This was Srinivasa Ramanujan. . .Professor Julian Huxley has referred to him as the Greatest Mathematician of the Century". Earlier Neville had in a radio broadcast claimed that "Srinivasa Ramanujan was a mathematician so great that his name transcends jealousies, the one superlatively great mathematician whom India has produced in the last thousand years." Neville might have forgotten Indian Greats like Bhaskara, but he was right as the Indian Mathematical Society hailed Ramanujan affirming that "by his unique mathematical talents and the amount of useful and original work, he has raised India in the estimation of the outside world.

But Ramanujan is not only for India. E. T. Bell called him *"a gift from heaven* : his all but supernatural insight into apparently unrelated formulas reveals hidden trails leading from one teritory to another and the analysts have now tasks provided for them in clearing the trails". On the other hand L. J. Mordell observes "In attempting to assess the standing of Ramanujan as a mathematician it is difficult not to be influenced by admiration and wonder at his success in becoming a professional mathematician inspite of the greatest difficulties. The estimate depends as for all mathematicians essentially upon the novelty and importance of his original work. . .To very few other mathematicians

are Klein's remarks, made many years ago, so appropriate as to Ramanujan : 'the secret of gifted productivity will always be that of finding new questions and new points of view, and without these mathematics has been advanced most by those who are distinguished more for intuition than for rigorous methods of proof."

And, how do well-wishers of India remember Ramanujan : J.B.S. Haldane, who on coming to India after Independence had exclaimed "Now, for the first time I have set foot on a free soil" has this to say :"It is no use celebrating Srinivasa Ramanujan unless we learn from his life. After five years of residence in India it is abundantly clear that Indian Public. Bodies and Universities have not done so. . .I am aware that he was offered a Chair in India after becoming a fellow of the Royal Society. But it is scandalous that India's great men should have *to wait for foreign recognition.* If Ramanujan's work had been recognised in India, he might never have emigrated and might be alive today." The same sentiment has been expressed by B.H. Newmann who emphasises that "It is important that India should encourage its native mathematical talent; if Ramanujan had been so encouraged as a young man, his life might have been happier and longer, his life and his impact on twentienth century mathematics greater."

All this advice from well-wishers came long ago. Censuswise the number of mathematicians in India has swelled. In the meantime we have remembered Ramanujan ritually time and again, but as an important Indian scientist points out" inspite of pouring money into this and that Institute, India has not produced one-tenth Ramanujan in forty years of Independence". Obviously, as long as the pinnacle of glory for a mathematician remains' to get recognition as an NRI (non-resident Indian) or to have the privilege of spending about half the time each year abroad, only a miracle may possibly change the situation. There is another danger which Nandy has correctly pointed out : "the semirural Kumbhakonam after producing the only great man in its long history, has relapsed into somnolence assimilating Ramanujan into a local myth only partly tainted by life." Let us read "India" in place of "Kumbhakonam and watch out.

1

Indian Mathematical Society and Ramanujan

Indian Mathematical Society was formed under the name "The Analytic Club". Its object was promotion of mathematical study and research in India. To start with the headquarter was located at Fergusson College, Pune. The Committee, which was to carry on the affairs of the Society consisted of

1. Prof. B. Hanumutha Rao, B.A. College of Engineering, Madras (President).
2. Prof. K.J. Sanjana, M.A., Samaldas College, Bhavanagar, (Treasurer).
3. Prof. R.P. Paranjpye, M.A. B Sc. Fergusson College, Pune (Librarian)
4. Mr. V. Ramaswami Iyer, M.A. Deputy Collector, Gooty (Secretary)
5. Prof. M.T. Narayaniengar, M.A., Central College, Bangalore (Secretary).

Besides the five office bearers there were 5 other members :

1. Mr. R. Ramachandra (Later Dewan Bahadur), B.A., Collector, Kurnool.
2. Prof. T.R. Venkataswami Naidu, B.A., Maharajah's College, Mysore.

3. Prof. R.N. Apte's M.A., LL.B. F.R.A.S., Rajaram College, Kolhapur,
4. Mr. Balak Ram, I.C.S., Deputy Accountant General, Allahabad,
5. Prof. A.C.C. wiltkinson,M.A., FR.A.S., Deccan College Pune

In August 1909, Prof. Devi Dayal, B.A. of D.A.V. College, Lahore was co-opted as an additional member.Membership spread over the present states of Tamil Nadu. Andhra, Karnataka, Kerala, Maharashtra, Gujarat and Undivided Punjab. In 1910 a draft Revised Constitution was circulated and adopted in 1911. Accordingly the Club got the name Indian Mathematical Society and the journal became the Journal Indian Mathematical Society.

The journal contained (a) Research articles (b) Questions and solutions, (c) Reviews (d) Short Notes. Also for many years the Journal contained a regular feature "Face of the Sky" by V. Ramesan, Vakil Madras High Court, giving astronomical readings covering two months between two issues of the Journal. What is important is that the articles in the journal upto 1913 covered areas of Geometry (Euclidean methods and Analytical), Calculus (Differential, Integral), Algebra (Computation like Theory of Equations, Determinants), Trigonometrical identities and equations, Statics, Dynamics and Partial Differential Equations etc., but very little Complex Analyis. Although the headquarters of the Society was at Pune and Paranjpye and Sarjana used to contribute in almost every issue of the Journal the bulk of the material came from the Southern Region, in particular Madras Presidency. Some people belonging to Undivided Punjab or Bombay Presidency who stayed at Calcutta or in the present-day Madhya Pradesh in connection with Government Service, retained their mathematical activity at their place of wurk also.

One of the great contributions that the Indian Mathematical Society and its journal made towards development of mathematics in India, of which the Society can rightly be proud is that it generously opened the pages of the journal and its functionaries like Ramachandra Rao, Ramaswami Iyer and Narayan Iyer provided ungrudging support and encouragement to Ramanujan. Ram-

anujan started writing in the journal in 1911. His first contribution was in the form of a brief problem which was communicated by P.V. Seshu Iyer. His first paper entitled "Some properties of Bernoulli's Numbers" also appeared in the same issue. In fact about one-third of his papers (11 out of a total of 37) were published in the Journal of Indian Mathematical Society—mostly before he left for Cambridge, only two appearing after his return. He made a significant contribution to the Question and Solutions Section of the journal. In all 46 questions submitted by him appeared out of which 34 were solved, 10 by Ramanujan himself, appearing in the form of solution in the journal or as part of some paper. There are still twelve unsolved problems as listed below.

1. Shew that

(i) $$\frac{1}{1-}\ \frac{e^{-2\pi}}{1+}\ \frac{e^{-4\pi}}{1+}\ \frac{e^{-6\pi}}{1+}$$

$$\cdots=[\sqrt{[\tfrac{1}{2}(5+\sqrt{5})]}\ -\ \tfrac{1}{2}(\sqrt{5}+1)]e^{\frac{2}{5}\pi},$$

(ii) $$\frac{1}{1-}\ \frac{e^{-\pi}}{1+}\ \frac{e^{-2\pi}}{1-}\ \frac{e^{-3\pi}}{1+}$$

$$\cdots=[\sqrt{[\tfrac{1}{2}(5-\sqrt{5})]}\ -\ \tfrac{1}{2}(\sqrt{5}-1)]e^{\frac{1}{5}\pi},$$

2. Shew that $$\frac{1}{e^{\cdot\pi}-1}\ +\frac{2}{e^{4\pi}-1}\ +\frac{3}{e^{6\pi}-1}\ +\cdots=\frac{1}{24}\ -\frac{1}{8\pi}$$

3. If n is positive shew that

$$\frac{1}{n}\ >\frac{1}{n+1}\ +\frac{1}{(n+2)^2}+\frac{3}{(n+3)^3}\ +\frac{4^2}{(n+4)^4}\ +\frac{5^2}{(n+5)^5}\ +\cdots;$$

and find approximately the difference when n is great. Hence shew that

$$\frac{1}{1001}\ +\ \frac{1}{1002^2}+\frac{3}{1003^3}+\frac{4}{1004^4}+\frac{5^3}{1005^5}\ +\cdots$$

is less than $\frac{1}{1000}$ by approximately 10^{-440}.

4. Let AB be a diameter and BC the chord of a circle ABC. Bisect the minor arc BC at M; and draw a chord BN equai to half of the chord BC, Join AM, Describe two circles with A and B as centresand AM and BN as radii, cutting each other at S and S', and cutting the given circle again at the points M and N' respectively. Join AN and BM intersecting at R, and also join AN' and BM' interesting at R'. Through B draw a tangent to the given circle, meeting AM and AM'produced at Q and Q' respectively. Produce AO and M'B to meet at P, and also produce AN' and MB to meet at P'. Shew that the eight points P, Q, R, S, S', R', Q', P's are cyclic, and that the circle passing through these eight points is orthogonal to the given circle ABC.

5. Shew that the roots of the equations

(i) $x^6-x^3+x^2+2x-1=0$,

(ii) $x^6+x^5-x^3-x^2-x+1=0$

can be expressed in terms of radicals.

6. Solve completely

$$x^2=a+y,\ y^2=a+z,\ z^2=a+u,\ u^2=a+x;$$

and deduce that, if $x=\sqrt{[5+\sqrt{[5+\sqrt{(5-\sqrt{(5+x)})}]}}$

then $a=1/2[2+\sqrt{5}+\sqrt{(15-6\sqrt{5})}]$;

and that, if $x=\sqrt{[5+\sqrt{[5-\sqrt{(5-+(5+x))}]}}$.

then $a=\frac{1}{4}[\sqrt{5}-2+\sqrt{(13-4\sqrt{5})}+\sqrt{[50+12=5-2=(65-20\sqrt{5})]}]$.

7, If $\phi(x)=e^{-x}+\frac{x}{1!}e^{-2x}+\frac{3x^2}{2!}e^{-3x}+\frac{4^2x^3}{3!}e^{-4x}+\frac{5^3x^4}{4!}e^{-5x}+\cdots$,

shew that $\phi(x)=1$ when x lies between 0 and 1; and that $\phi(x)\neq 1$ when $x>1$. Find the limit of

$$[\phi(1+\epsilon)-\phi(1)]/\epsilon$$

as $\epsilon\to 0$ through positive values.

8. Let p be the perimeter and e the eccentricity of an ellipse whose centre is C, and let CA and CB be a semi-major and semi-minor axis. From CA cut off CQ equal to CB, and also produce

AC to P making CP equal to CB. From A draw AN perpendicular to CA (in the direction of CB). From Q draw QM making with QA an angle equal to ϕ (which is to be determined) and meeting AN at M. Join PM and draw PN making with PM an angle equal to half of the angle APM, and meeting AN at N. With P as centre and PA as radius describe a circle, cutting PN at K, and meeting PB produced at L. Then if

$$\frac{\text{arc AL}}{\text{arc AK}} = \frac{\phi}{4\text{AN}},$$

trace the changes in ϕ when e varies from 0 to 1. In particular, shew that $\phi = 30°$ when $e = 0$; $\phi = 30°$ when $e = 1$; $\phi = 30^0$ when $e = 0.99948$ nearly; ϕ assumes the minimum value of about $29°58\frac{3'}{4}$ when e is about 0.999886; and ϕ assumes the maximum value of about $30°44\frac{1}{4}$ when e is about 0.9589.

9. If δ_n denote the number of divisors of n (e. g. $\delta_1=1$, $\delta_2=2$, $\delta_3=3$,...) shew that

$$\text{(i)}\quad \delta_1 - \frac{1}{3}\delta_2 + \frac{1}{5}\delta_5 - \frac{1}{7}\delta_7 + \frac{1}{9}\delta_9 - \ldots$$

is a convergent series; and that

$$\text{(ii)}\quad \delta_1 - \frac{1}{2}\delta_2 + \frac{1}{3}\delta_3 - \frac{1}{4}\delta_4 + \frac{1}{5}\delta_5 - \ldots$$

is a divergent series in the strict sense (i. e. not oscillating).

10. If F (x) denotes the fractional part of x (e.g. $F(\pi) = 0.14159....$) and if N is a positive integer, shew that

$$\text{(i)}\quad \lim_{N\to\infty} NF(N\sqrt{2}) = \frac{1}{2\sqrt{2}}, \quad \lim_{N\to\infty} NF(N\sqrt{3}) = \frac{1}{\sqrt{3}}$$

$$\lim_{N\to\infty} NF(N\sqrt{5}) = \frac{1}{2\sqrt{5}}, \quad \lim_{N\to\infty} NF(N\sqrt{6}) = \frac{1}{\sqrt{6}},$$

$$\lim_{N\to\infty} NF(N\sqrt{7}) = \frac{3}{2\sqrt{7}};$$

$$\text{(ii)}\quad \lim_{N\to\infty} N(\log N)^{1-p} F(Ne^{2/n}) = 0,$$

2

Some Properties of Bernoulli's Numbers

(*Journal of the Indian Mathematical Society*, III, 1911, 219–234)

1. Let the well-known expansion of x cot x (vide Edward's *Differential Calculus*, § 149) be written in the form

$$x \cot x = 1 - \frac{B_2}{2!}(2x)^2 - \frac{B_4}{4!}(2x)^4 - \frac{B_6}{6!}(2x)^6 - \ldots, \quad \ldots(1)$$

from which we infer that B_0 may be supposed to be -1.

$$\text{Now } \cot x = \frac{\cos x}{\sin x} = \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}$$

$$= \frac{\sin 2x}{1 - \cos 2x} = \frac{\frac{2x}{1!} - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots}{\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \ldots}$$

$$= \frac{1 + \cos 2x}{\sin 2x} = \frac{2 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots}{2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots}$$

where n is any integer and p is any positive number.

11. Shew that

(i) $$\int_0^\infty \frac{\sin nx\,dx}{x+\cfrac{1}{x+}\ \cfrac{2}{x+}\ \cfrac{3}{x+}\ \ldots.} = \frac{\sqrt{(\frac{1}{2}\pi)}}{\cfrac{1}{n+}\ \cfrac{2}{n+}\ \cfrac{3}{n+}\ \ldots}$$

(ii) $$\int_0^\infty \frac{\sin \frac{1}{2}(\pi nx)\,dx}{x+\cfrac{1^2}{x+}\ \cfrac{2^2}{x+}\ \cfrac{3^2}{x+}\ \ldots} = \frac{1}{n+}\ \frac{1^2}{n+}\ \frac{2^2}{n+}\ \frac{3^2}{n+}\ \ldots$$

12. Shew that (i) $(7^3\sqrt{20}-19)^{\frac{1}{3}} = \sqrt[3]{\frac{5}{3}} - \sqrt[3]{\frac{2}{3}}$;

(ii) $$\left(4\sqrt[3]{\frac{2}{3}} - 5\sqrt[3]{\frac{1}{3}}\right)^{\frac{1}{6}} = \sqrt[3]{\left(\frac{4}{9}\right)} - \sqrt[3]{\left(\frac{2}{9}\right)} + \sqrt[3]{\frac{1}{9}}.$$

Multiplying both sides in each of the above three relations by the denominator of the right-hand side and equating the coefficients of x^n on both sides, we can write the results thus :

$$c_1 \frac{B_{n-1}}{2} - c_3 \frac{B_{n-3}}{2^3} + c_3 \frac{B_{n-5}}{2^5} - \ldots + \frac{(-1)^{\frac{1}{2}(n-1)}}{2^n} B_0 + \frac{n}{2^n} (-1)^{\frac{1}{2}(n-1)} = 0, \quad \ldots(2)$$

where n is any odd integer;

$$c_2 B_{n-2} - c_4 B_{n-4} + c_6 B_{n-6} - \ldots + (-1)^{\frac{1}{2}(n-2)} B_0 + \frac{n}{2} (-1)^{\frac{1}{2}(n-2)} = 0, \quad \ldots(3)$$

where n is any even integer;

$$c_1 B_{n-1} - c_3 B_{n-3} + c_5 B_{n-5} - \ldots + (-1)^{\frac{1}{2}(n-1)} B_0 + \frac{n}{2} (-1)^{\frac{1}{2}(n-1)} = 0, \quad \ldots(4)$$

where n is any odd integer greater than unity.

From any one of (2), (3), (4) we can calculate the B's. But as n becomes greater and greater the calculation will get tedious. So we shall try to find simpler methods.

2. We know $(x \cot x)^2 = -x^2 \left(1 + \frac{d(\cot x)}{dx}\right)$.

Using (1) and equating the coefficients of x^n on both sides, and simplifying, we have

$$\tfrac{1}{2}(n+1) B_n = c_2 B_2 B_{n-2} + c_4 B_4 B_{n-4} + c_6 B_6 B_{n-6} + \ldots,$$

the last term being $c_{\frac{1}{2}(n-1)} B_{\frac{1}{2}(n-1)} B_{\frac{1}{4}n+1}$

or $\frac{1}{2} c_{\frac{1}{2}n} (B_{\frac{1}{2}n})^2$ according as $\frac{1}{2}n$ is odd or even. $\ldots(5)$

A similar result can be obtained by equating the coefficients of x^n in the identity;

$$\frac{d \tan x}{dx} = 1 + \tan^2 x.$$

3. Again

$$-\tfrac{1}{2}x(\cot \tfrac{1}{2}x + \cot h\ \tfrac{1}{2}x) = -\tfrac{1}{2}x(\cot \tfrac{1}{2}x + i \cot \tfrac{1}{2} ix)$$

$$= 2\{B_0 + B_4 \frac{x^4}{4!} + B_8 \frac{x^8}{8!} + \ldots\},$$

by using (1), The expression may also be written

$$-\tfrac{1}{2}x \frac{(\cos \tfrac{1}{2}x \sin \tfrac{1}{2}ix + i \sin \tfrac{1}{2}x \cos \tfrac{1}{2}ix)}{\sin \tfrac{1}{2}x \sin \tfrac{1}{2}ix}$$

$$= -\tfrac{1}{2}x \frac{(1+i) \sin \tfrac{1}{2}x\,(1+i) - (1-i) \sin \tfrac{1}{2}x\,(1-i)}{\cos \tfrac{1}{2}x\,(1-i) - \cos \tfrac{1}{2}x\,(1+i)}$$

$$= -x \frac{\dfrac{x}{1!} - \dfrac{x^5}{2^2.5!} + \dfrac{x^9}{2^4.9!} - \ldots}{\dfrac{x^2}{2!} - \dfrac{x^6}{2^2.6!} + \dfrac{x^{10}}{2^4.10!} - \ldots}$$

by expanding the numerator and the denominator, and simplifying by De Moivre's theorem.

$$\text{Hence } 2\left(B_2 + B_4 \frac{x^4}{4!} + B_8 \frac{x^8}{8!} + \ldots\right) = -x \frac{\dfrac{x}{1!} - \dfrac{x^5}{2^2.5!} + \ldots}{\dfrac{x^2}{2!} - \dfrac{x^6}{2^2.6!} + \ldots} \qquad \ldots(6)$$

Similarly

$$-\frac{1}{2}x\left(\cot \frac{1}{2}x - \coth \frac{1}{2}x\right) = 2\left(B_2 \frac{x^2}{2!} + B_6 \frac{x^6}{6!} + B_{10} \frac{x^{10}}{10!} + \ldots\right)$$

$$= x \frac{\tfrac{1}{2}(1-i) \sin \tfrac{1}{2}x\,(1+i) - \tfrac{1}{2}(1+i) - \tfrac{1}{2}(1+i) \sin \tfrac{1}{2}x\,(1-i)}{\cos \tfrac{1}{2}x\,(1+i) - \cos \tfrac{1}{2}x\,(1-i)}$$

$$= x \frac{\frac{x^3}{2.3!} + \frac{x^7}{2^3.7!} + \frac{x^{11}}{2^5.11!} + \ldots}{\frac{x^2}{2!} + \frac{x^6}{2^2.6!} + \frac{x^{10}}{2^4.10!} - \ldots}. \qquad \ldots(7)$$

Proceeding as in § 1 we have, if n is an even integer greater than 2.

$$c_2 \frac{B_{n-2}}{2} - c_6 \frac{B_{n-6}}{2^3} + c_{10} \frac{B_{n-10}}{2^5} - \ldots + \frac{n}{2^{\frac{1}{2}(n+2)}} (-1)^{\frac{1}{4}n} \text{ or}$$

$$\frac{n}{2^{\frac{1}{4}(n+2)}} (-1)^{\frac{1}{4}(n-2)} = 0, \qquad \ldots(8)$$

according as n or n–2 is a multiple of 4.

Analogous results can be obtained from $\tan \frac{1}{2}x \pm \tanh \frac{1}{2}x$.

In (2), (3) and (4) there are $\frac{1}{2}n$ terms, while in (5) and (8) there are $\frac{1}{4}n$ or $\frac{1}{4}(n-2)$ terms. Thus B can be found from only half of the previous B's.

4. A still simpler method can be deduced from the following identities. If 1, ω, ω^2 be the three cube roots of unity, then

$$4 \sin x \sin x\omega \sin x\omega^2 = -(\sin 2x + \sin 2x\omega + \sin 2x\omega^2),$$

as may easily be verified.

By logarithmic differentiation, we have

$$\cot x + \omega \cot x\omega + \omega^2 \cot x\omega^2 = 2 \frac{\cos 2x + \omega \cos 2x\omega + \omega^2 \cos 2x\omega^2}{\sin 2x + \sin 2x\omega + \sin 2x\omega^2}$$

Writing $\frac{1}{2}x$ for x,

$$-\tfrac{1}{2}x (\cot \tfrac{1}{2}x + \omega \cos \tfrac{1}{2}x\omega + \omega^2 \cot \tfrac{1}{2}x\omega^2) =$$

$$-x \frac{\cos x + \omega \cos x\omega + \omega^2 \cos x\omega^2}{\sin x + \sin x\omega + \sin x\omega^2},$$

and proceeding as in 3, we get

$$3\left(B_0+B_6\frac{x^6}{6!}+B_{12}\frac{x^{12}}{12!}+\ldots\right)=-x\,\frac{\frac{x^2}{2!}-\frac{x^8}{8!}+\frac{x^{14}}{14!}-\ldots}{\frac{x^3}{3!}-\frac{x^9}{9!}+\frac{x^{15}}{15!}-\ldots}. \quad \ldots(9)$$

Again

$$\cot\tfrac{1}{2}x\omega-\cot\tfrac{1}{2}x\omega^2=2\frac{\cos x\omega^2-\cos x\omega}{\sin\tfrac{1}{2}x\sin\tfrac{1}{2}x\omega\sin\tfrac{1}{2}x\omega^2}$$

$$=\frac{2(\cos x\omega-\cos x\omega^2)}{\sin x+\sin x\omega+\sin x\omega^2}.$$

Multiplying both sides by $-\frac{1}{2}x(\omega^2-\omega)$ and adding to the corresponding sides of the previous result, we have

$$-\tfrac{1}{2}x(\cot\tfrac{1}{2}x+\omega^2\cot 2x\omega+\omega\cot\tfrac{1}{2}x\omega^2)=$$

$$-x\,\frac{\cos x+\omega^2\cos x\omega+\omega\cos x\omega^2}{\sin x+\sin x\omega+\sin x\omega^2}.$$

Hence, as before,

$$3\left(B_2\frac{x^2}{2!}+B_8\frac{x^8}{8!}+B_{14}\frac{x^{14}}{14!}+\ldots\right)=x\,\frac{\frac{x^4}{4!}-\frac{x^{10}}{10!}+\frac{x^{16}}{10!}-\ldots}{\frac{x^3}{3!}-\frac{x^9}{9!}+\frac{x^{15}}{15!}-\ldots}. \quad (10)$$

Similarly

$$-(\cot\tfrac{1}{2}x+\cot\tfrac{1}{2}x\omega+\cot\tfrac{1}{2}x\omega^2)=x\,\frac{\cos x+\cos x\omega+\cos x\omega^2-3}{\sin x+\sin x\omega+\sin x\omega},$$

and therefore,

$$6\left(B_4\frac{x^4}{4!}+B_{10}\frac{x^{10}}{10!}+B_{16}\frac{x^{16}}{16!}+\ldots\right)=x\,\frac{\frac{x^6}{6!}-\frac{x^{12}}{12!}+\frac{x^{18}}{18!}-\ldots}{\frac{x^3}{3!}-\frac{x^9}{9!}+\frac{x^{15}}{15!}-\ldots}. \quad (11)$$

Multiplying up and equating coefficients in (9), (10) and (11) as usual, we have,

$$c_3B_{n-3}-c_9B_{n-9}+c_{15}B_{n-15}-\ldots=0, \quad \ldots(12)$$

the last term being $\frac{1}{6}n(-1)^{\frac{1}{6}(n-1)}$, $\frac{1}{3}n(-1)^{\frac{1}{6}(n+1)}$,

$$\text{or } \frac{1}{3}n(1-)^{\frac{1}{6}(n-3)}.$$

Again, dividing both sides in (10) by x and differentiating, we have

$$3\left(B_2\frac{1}{2;}+7B_8\frac{x^6}{8;}+13B_{14}\frac{x^{12}}{14;}+\ldots\right)=\frac{d}{dx}\left(\frac{\frac{x^4}{4;}+\frac{x^{10}}{10;}+\frac{x^{16}}{16;}-\ldots}{\frac{x^3}{3;}+\frac{x^9}{9;}+\frac{x^{15}}{15;}-\ldots}\right)$$

$$=1-\frac{\frac{x^2}{2!}+\frac{x^8}{8!}+\frac{x^{14}}{14!}-\ldots}{\frac{x^3}{3!}-\frac{x^9}{9!}+\frac{x^{15}}{15!}-\ldots}\cdot\frac{\frac{x^4}{4!}-\frac{x^{10}}{10!}+\frac{x^{16}}{16!}-\ldots}{\frac{x^3}{3!}-\frac{x^9}{9!}+\frac{x^{15}}{15!}-\ldots}.$$

Hence by (9) and (10),

$$3\left(B_2\frac{x^2}{2;}+7B_8\frac{x^8}{8;}+13B_{14}\frac{x^{14}}{14;}+\ldots\right)$$

$$=x^2+9\left(B_0+B_6\frac{x^6}{6!}+B_{12}\frac{x^{12}}{12!}+\ldots\right)$$

$$\left(B_2\frac{x^2}{2!}+B_8\frac{x^8}{8!}+B_{14}\frac{x^{14}}{14;}+\ldots\right).$$

Equating the coefficients of x^n we have, if $n>2$ and $n-2$ is a multiple of 6,

$$\frac{1}{3}(n+2)B_n=c_6B_{n-6}B_6+c_{12}B_{n-12}B_{12}+c_{18}B_{n-18}B_{18}+\qquad\ldots(33)$$

From (12) the B's can be calculated very quickly and (13) may prove useful in checking the calculations. The number of terms is one-third of that in (4); thus B_{24} is found from B_{18}, B_{12} and B_6.

5. We shall see later on how the B's can be obtained from their properties only. But to know these properties, it will be

convenient to calculate a few B's by substituting 3,5,7,9,..., for n in succession in (12). Thus

$$B_0=-1;\ B_2=\frac{1}{6};\ B_4=\frac{1}{30};\ B_6=\frac{1}{42};\ B_8-\frac{1}{3}B_2=-\frac{1}{45};$$

$$B_{10}-\frac{5}{2}B_4=-\frac{1}{132};\ B_{12}-11B_6=-\frac{4}{455};\ B_{14}-\frac{143}{8}B_8+\frac{B_2}{5}=\frac{1}{120};$$

$$B_{16}-\frac{286}{3}B_{10}+4B_4=\frac{1}{306};\ B_{18}-221\,B_{12}+\frac{204}{5}B_6=\frac{3}{665};$$

$$B_{20}-\frac{3230}{7}B_{14}+\frac{1938}{7}B_8-\frac{B_2}{7}=-\frac{1}{231};$$

$$B_{22}-\frac{3553}{4}B_{16}+\frac{7106}{5}B_{10}-\frac{11}{2}B_4=-\frac{1}{552};$$

and so on. Hence we have finally the following values :

$$B_2=\frac{1}{6};\ B_4=\frac{1}{30};\ B_6=\frac{1}{42};\ B_8=\frac{1}{30};\ B_{10}=\frac{5}{66};\ B_{12}=\frac{691}{2730};$$

$$B_{14}=\frac{7}{6};\ B_{16}=\frac{3617}{510};\ B_{18}=\frac{43867}{798};\ B_{20}=\frac{174611}{330};\ B_{22}=\frac{854513}{138};$$

$$B_{24}=\frac{236364091}{2730};\ B_{26}=\frac{8553103}{6};\ B_{28}=\frac{23749461029}{870};$$

$$B_{30}=\frac{8615841276005}{14322};\ B_{32}=\frac{7709321041217}{510};\ B_{34}=\frac{2577687858367}{6};$$

$$B_{36}=\frac{26315271553053477373}{1919190};\ B_{38}=\frac{29299939138841559}{6};$$

$$B_{40}=\frac{261082718496449122051}{13530};\ \ldots,\qquad B_{\infty}=\infty.$$

6. It will be observed* that, if n is even but not equal to zero.

(i) B_n is a fraction and the numerator of B_n/n in its lowest terms is a prime number, ...(14)

(ii) the denominator of B_n contains each of the factors 2 and 3 once and only once, ...(15)

(iii) $2^n (2^n-1) B_n/n$ is an integer and consequently $2 (2^n-1) B_n$ is an odd integer. ...(16)

From (16) it can easily be shewn that the denominator of $2 (2^n-1) B_n/n$ in its lowest terms is the greatest power of 2 which divides n; and consequently; if n is not a multiple of 4, then $4 (2^n-1)B_n/n$ is an odd integer. ...(17)

It follows from (14) that the numerator of B_n in its lowest terms is divisible by the greatest measure of n prime to the denominator, and the quotient is a prime number ...(18)

Examples : (a) 2 and 3 are the only prime factors of 12, 24 and 36 and they are found in the denominators of B_{12}, B_{24} and B_{36} and their numerators are prime numbers.

(b) 11 is not found in the denominator of B_{22}, and hence its numerator is divisible by 11 similarly, the numerators of B_{26}, B_{34}, B_{38} are divisible by 13, 17, 19, respectively, and the quotients in all cases are prime numbers.

(c) 5 is found in the denominator of B_{20} and not in that of B_{30}, and consequently the numerator of B_{30} is divisible by 5 while that of B_{20} is a prime number. Thus we may say that if a prime number appearing in n is not found in the denominator it will appear in the numerator, and vice versa.

7. Next, let us consider the denominators.

All the denominators are divisible by 6; those of B_4, B_8, B_{x2}... by 5; those of B_6, B_{12}, B_{18},... by 7; those of B_{10}, B_{20}, B_{30}, ... by 11; but those of B_8, B_{16}, B_{24},.... are not divisible by 9: and those of B_{14}, B_{28},... are not divisible by 15. Hence we may infer that :

*See § 12 below.

the denominator of B_n is the continued product of prime numbers which are the next numbers (in the natural order) to the factors of n (including unity and the number itself). ...(19)

As an example take the denominator of B_{24}. Write all the factors of 24, viz. 1,2,3,4,6,8,12,24. The next numbers to these are 2,3,5,7,9,13,25. Strike out the composite numbers and we have the prime numbers 2,3,5,7,13. And the denominator of B_{24} is the product of 2,3,5,7,13, i. e. 2730.

It is unnecessary to write the odd factors of n except unity, as the next numbers to these are even and hence composite.

The following are some further examples :

Even factors of n and unity	Denominator of B_n
B_2 ... 1,2, ...	2.3=6
B_6 ... 1,2,6, ...	2.3.7=42
B_{12} ... 1,2,4,6,12	2.3.5.7.13=2730
B_{20} ... 1,2,4,10,20	2.3.5.11.=330
B_{30} ... 1,2,6,10,30	2.3.7.11.31=14322
B_{42} ... 1,2,6,14,42	2.3.7.43=1806
B_{56} ... 1,2,4,8,14,28,56 ...	2.3.5.29=870
B_{72} ... 1,2,4,6,8,12,18,24,36,72 ...	2.3.5.7.13.19.37.73= 140100870
B_{90} ... 1,2,6,10,18,30,90	2.3.7.11.19.31=272118
B_{110}... 1,2,10,22,110	2.3.11.23=1518

8. Again taking the fractional part of any B and splitting it into partial fractions, we see that :

$$\text{the fractional part of } B_n = (-1)^{\frac{1}{2}n}\left\{\text{the sum of the reciprocals of the prime factors of the denominator of } B_n\right\} - (-1)^{\frac{1}{2}n}. \quad \ldots (20)$$

Thus the fractional part of $B_{16} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} - 1 = \frac{47}{510}$;

that of $B_{22} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{23} = \frac{17}{138}$;

that of $B_{28} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{29} - 1 = \frac{59}{870}$;

and so on.

9. It can be inferred from (20) that :

if G be the G.C.M. and L the L.C.M. of the denominators of B_m and B_n, then L/G is the denominator of $B_m - (-1)^{\frac{1}{2}(m-n)} B_n$, and hence, if the denominators of B_m and B_n are equal, then $B_m - (-1)^{\frac{1}{2}(m-n)} B_n$ is an integer. ...(21)

Example : $B_{24} - B_{12}$ and $B_{32} - B_{16}$ are integers, while the denominator of $B_{10} + B_{20}$ is 5.

It will be observed that :

(1) if n is a multiple of 4, then the numerator of $B_n - \frac{1}{30}$ in its lowest terms is divisible by 20 ; but if n is not a multiple of 4 then that of $\frac{B_n}{n} - \frac{1}{12}$ in its lowest terms is divisible by 5 :, ...(22)

(2) if n is any integer, then

$$2(2^{4n+2} - 1)\frac{B_{4n+2}}{2n+1}, \quad 2(2^{8n+4} - 1)\frac{B_{8n+4}}{2n+1},$$

$$2(2^{8n+4} - 1), \frac{B_{16n+8}}{2n+1}$$

are integers of the form 30 p + 1. (...23)

10. If a B is known to lie between certain limits, then it is possible to find its exact value from the above properties.

Suppose we know that B_{22} lies between 6084 and 6244 ; its exact value can be found as follows.

The fractional part of $B_{22} = \frac{17}{138}$ by (20), also B_{22} is divisible by 11 by (18). And by(22) $B_{22} - \frac{11}{6}$ must be divisible by 5. To satisfy these conditions B_{22} must be either $1637\frac{17}{138}$ or $6192\frac{17}{138}$.

But according to (18) the numerator of B_{22} should be a prime number after it is divided by 11; and consequently B_{22} must be equal to $6192\frac{17}{138}$ or $\frac{854513}{138}$, since the numerator of $6137\frac{17}{138}$ is divisible not only by 11 but also by 7 and 17.

11. It is known (Edward's Differential Calculus, Ch. V, Ex. 29) that

$$B_n = \frac{2.n!}{(2\pi)^n}\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\ldots\right),$$

or
$$\frac{2.n!}{(2\pi)^n} = B_n\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right) \qquad \ldots(24)$$

where 2,3,5,...are prime numbers.

Also
$$\frac{B_n}{2n} = \int_0^\infty \frac{x^{n-1}}{e^{2nx}-1}\,dx. \qquad \ldots(25)$$

For
$$\int_0^\infty \frac{x^{n-1}}{e^{2\pi x}-1}\,dx = \int_0^\infty x^{n-1}\left(e^{-2\pi x}+e^{-4\pi x}+\ldots\right)dx$$

$$= \frac{(n-1)}{(2\pi)^n}\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\ldots\right) = \frac{B_n}{2n}$$

$$= -\int_0^\infty \frac{x^{2p-2}\log(1-e^{-2\pi nx})}{\pi(1+x^2)}\,dx.$$

We can find the integral part of B_n, and since the fractional part can be found, as shown in §8, the exact value of B_n is known. Unless the calculation is made to depend upon the values of $\log_{10}e$, $\log_e 10, \pi, \ldots$, which are known to a great number of decimal places, we should have to find the logarithms of certain numbers whose values are not found in the tables to as many places of decimals as we require. Such difficulties are removed by the method given in §13.

12. Result (14) to (17), (20) and (21) can be obtained as follows. We have

$$\frac{1}{2x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\frac{1}{(x+4)^2}+\ldots$$
$$-\frac{1}{x}-\frac{1}{6(x^2-x)}+\frac{1}{5(x^5-x)}+\frac{1}{7(x^7-x)}+\frac{1}{11(x^{11}-x)}+\ldots$$
$$=\frac{1}{15x}-\frac{7}{17x}+\frac{55}{19x}+\frac{529}{21x}+\ldots, \qquad \ldots(28)$$

where 5, 7, 11, 13, are prime numbers above 3. If we can prove that the left-hand side of (28) can be expanded in ascending powers of $1/x$ with integral coefficients, them (20) and (21) are at once deduced as follows.

From (27) we have

$$\frac{d^2\log\Gamma(n+1)}{dn^2}=\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\ldots$$
$$=\frac{1}{n}-\frac{1}{2n^2}+\frac{B_2}{n^3}-\frac{B_4}{n^5}+\frac{B_6}{n^7}-\frac{B_8}{n^9}+\ldots$$
$$-(-1)^p\frac{B_{2p}\,\theta}{n^{2p+1}}, \qquad \ldots(29)$$

by (24). In a similar manner

$$\int_0^\infty \frac{x^n}{(e^{\pi x}-e^{-\pi x})^2}\,dx=\frac{B_n}{4\pi},$$

and $$\int_0^\infty x^{n-2}\log(1-e^{-2\pi x})\,dx=-\frac{\pi B_n}{n(n-1)}. \qquad ...(26)$$

Take logarithms of both sides in (24) and write for $\log_e n!$ the well-known expanion $\log_e \Gamma(n+1)$, as in Carr's *Synopsis*, viz.

$$(n+\tfrac{1}{2})\log n-n+\tfrac{1}{2}\log 2\pi+\frac{B_2}{1.2n}-\frac{B_4}{3.4n^3}+\frac{B_6}{5.6n^5}$$

$$-(-1)^p\frac{B_{2p}\,\theta}{(2p-1)2pn^{2p-1}} \qquad ...(27)$$

where $0<\theta<1$, and where

$$\frac{B_{2p}\,\theta}{(2p-1)\,2pn^{2p-1}}=\frac{B_{2p}}{(2p-1)2pn^{2p-1}}-\frac{B_{2p+2}}{(2p+1)\,(2p+2)n^{2p+1}}+\ldots$$

$$=-\frac{1}{\pi}\int_0^\infty \frac{x^{2p-2}}{n^{2p-1}}\log(1-e^{-2\pi x})\,dx+\frac{1}{\pi}\int_0^\infty \frac{x^{2p}}{n^{2p+1}}\log(1-e^{-2\pi x})\,dx.$$

$$=-\frac{1}{\pi}\int_0^\infty \left(\frac{x^{2p-2}}{n^{2p-1}}-\frac{x^{2p}}{n^{2p+1}}+\ldots\right)\log(1-e^{-2\pi x})\,dx$$

$$=-\frac{1}{\pi}\int_0^\infty \frac{x^{2p-2}}{n^{2p-3}\,(n+x)^2}\log(1-e^{-2\pi x})\,dx$$

where

$$-\frac{B_{2p}\,\theta}{n^{2p+1}} = \frac{B_{2p}}{n^{2p+1}} - \frac{B_{2p+2}}{n^{2p+2}} + \dots$$

$$= 4\pi \int_0^\infty \frac{x^{2p}}{n^{2p+1}\,(e^{\pi x} - e^{-\pi x})^2}\,dx$$

$$-4\pi \int_0^\infty \frac{x^{2p+2}}{n^{2p+3}\,(e^{\pi x} - e^{-\pi x})^2}\,dx \quad \dots$$

$$= \pi \int_0^\infty \left\{ \frac{x^{2p}}{n^{2p+1}} - \frac{x^{2p+2}}{n^{2p+3}} + \dots \right\} \frac{dx}{\sinh^2 \pi x}$$

$$= \pi \int_0^\infty \frac{x^{2p}}{n^{2p-1}\,(n^2+x^2)} \cdot \frac{dx}{\sinh \pi x}$$

$$= \int_0^\infty \frac{(\pi x)^{2p}}{(1+x^2)\,\sinh^2 \pi n x}\,dx$$

Substituting the result of (29) in (28) we see that

$$\frac{B_2}{x^3} - \frac{B_4}{x^5} + \frac{B_6}{x^7} - \dots - \frac{1}{6(x^3 - x)} + \frac{1}{5(x^5 - x)} + \frac{1}{7(x^7 - x)} + \frac{1}{11(x^{11} - x)} + \dots$$

where 5, 7, 11, ... are prime numbers, can be expanded in ascending powers of $1/x$ with integral coefficients.

Therefore $B_2 - \frac{1}{6}$, $-B_4 - \frac{1}{6} + \frac{1}{5}$, $B_6 - \frac{1}{5} + \frac{1}{7}$, $-B_8 - \frac{1}{6} + \frac{1}{5}$, $B_{10} - \frac{1}{6} + \frac{4}{11}$, . . . which are the coefficients of $1/x^3$, $1/x^5$, $1/x^7$, . . ., are integers.

Writing $\frac{1}{2} + \frac{1}{3} - 1$ for $-\frac{1}{6}$ we get the results of (20) and (21).

Again changing n to $\frac{1}{2}n$ in (29), and subtracting half of the result from (29), we have

$$\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}-\ldots$$
$$=\frac{1}{2n^2}-\frac{(2^2-1)B_2}{n^3}+\frac{(2^4-1)B_4}{n^5}-\frac{(2^6-1)B_6}{n^7}$$
$$\ldots+(-1)^p\,(2^{2p}-1)\,\frac{B_{2p}\,\theta}{n^{2p+1}},\qquad\ldots(30)$$

where $0<\theta<1$, and also, by (29),

$$\binom{2p}{2}-1\Big)\frac{B_{2p}\,\theta}{n^{2p+1}}=\int_0^\infty\frac{\pi x^{2p}\cosh \pi nx}{(1+x^2)\sinh^2\pi nx}\,dx.$$

Thus we see that, if we can prove that twice the left-hand side of (30) can be expanded in ascending powers of 1/n with integral co-efficients, the second part of (16) is at once proved.

Again from (27) we have

$$\frac{d\log\Gamma(n+1)}{dn}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}-\gamma$$
$$=\log n+\frac{1}{2n}-\frac{B_2}{2n^2}+\frac{B_4}{4n^4}-\frac{B_6}{6n^6}+\frac{B_8}{8n^8}-\ldots+$$
$$(-1)^p\,\frac{B_{2p}\theta}{2pn^{2p}},\qquad\ldots(31)$$

where $0<\theta<1$; and also, by (25),

$$\frac{B_{2p}\theta}{2pn^{2p}}=\int_0^\infty\frac{2x^{2p-1}}{(1+x^2)\,(e^{2\pi nx}-1)}\,dx,$$

from which it can easily be shown that

$$\frac{1}{n+2}-\frac{1}{n+4}+\frac{1}{n+6}-\frac{1}{n+8}+\frac{1}{n+10}-\cdots$$

$$=\frac{1}{2n}-2(2^2-1)\frac{B_2}{2n^2}+2^3(2^4-1)\frac{B_4}{4n^4}-2^5(2^6-1)\frac{B_6}{6n^6}+\ldots$$

$$+(-1)^p\ 2^{2p-1}\ (2^{2p}-1)\frac{B_{2p}\theta}{2pn^{2p}}-\cdots, \qquad \ldots(32)$$

where $0<\theta<1$; and also, by (31),

$$2^{2p-1}\ (2^{2p}-1)\frac{B_{2p}\theta}{2pn^{2p}}=\int_0^\infty \frac{x^{2p-1}}{2(1+x^2)\sinh\frac{1}{2}(\pi nx)}\,dx.$$

From the above theorem we see that, if we can prove that

$$2\left(\frac{1}{n+2}-\frac{1}{n+4}+\frac{1}{n+6}-\ldots\right),$$

can be expanded in ascending powers of $1/n$ with integral coefficients, then the first part of (16) at once follows.

13. The first few digits, and the number of digits in the integral part as well as in the numerator of B_n, can be found from the approximate formula :

$$\log_{10}B_n=(n+\tfrac{1}{2})\log_{10}n-1.2324743503n+0.700120,$$

the true value being greater by about $0.0362/n$ when n is large ...(33)

This formula is proved as follows; taking logarithms of both sides in (24),

$$\log_e B_n=(n+\tfrac{1}{2})\log_e n-n(1+\log_e 2\pi)+\tfrac{1}{2}\log_e 8\pi$$

nearly. Multiplying both sides by $\log_{10}e$ or .4342944819, and reducing, we can get the result.

14. Changing n to $n-2$ in (24) and taking the ratio of the two results, we have

$$B_n=\frac{n(n-1)}{4\pi^2}B_{n-2}\left(1-\frac{2^2-1}{2^n-1}\right)\left(1-\frac{3^2-1}{3^n-1}\right)\left(1-\frac{5^2-1}{5^n-1}\right)\ldots, \qquad \ldots(34)$$

where 2, 3, 5,.... are prime numbers.

15. From the preceding theorems we know some of the properties of B_n for all positive even values of n. As an example let us take $B_{444} = N/D$.

The fractional part of B_{444} is $\frac{23975417}{90709710}$ by (20). The numerator of B_{444} is divisible by 37 and the quotient is a prime number by (18). Again $\log_{10} B_{444} = 630.2433$, nearly, by (33). Therefore the integral part of B_{444} contains 631 digits, the first 4 digits being 1751. Again

$$\log_{10} N = \log_{10} B_{444} + \log_{10} D = 630.2433 + \log_{10} 90709710 =$$

638.2010 nearly. Therefore N contains 639 digits, the first four digits being 1588.

Again the numerator of $B_{444} - \frac{1}{30}$ is divisible by 20; that is to say, if $\left[B_{444}\right]$ is the integral part of B444' $\left[B_{444}\right] + \frac{23975417}{90709710} - \frac{1}{30} = \left[B_{444}\right] + \frac{698392}{3023657}$ has a numerator divisible by 20. Therefore the integral part of B_{444} ends with 4 and also the figure next to the last is even.

Hence N ends with 57 and also the third figure from the last is even.

16. Instead of starting with cot x as in §§ 2 and 3, we may start with tan x or cosec x and get other similar results.

Thus :

$$\text{(i)}\ \frac{4}{3} B_n (2^n - 1) = c_6 B_{n-6} (2^{n-6} - 1) - c_{12} B_{n-12} (2^{n-12} - 1) + \ldots$$

$$+ \frac{1}{3^n} (-1)^{\frac{1}{6}(n-2)} \text{ or } \frac{1}{6^n} (-1)^{\frac{1}{6}(n-4)} \text{ or } \frac{1}{3} n (-1)^{\frac{1}{6}(n-6)}$$

...(36)

Hence we see that $\frac{B_n}{B_{n-2}}$ approaches $\frac{n(n-1)}{4\pi^2}$ very rapidly as n becomes greater and greater ...(35)

From the value of π, viz. 3.14159,26535,89793,23846, 26433,83279,50288,41971,69399,..., the integral part of any B can be found from the previous B;, and from the integral part the exact value can at once be written by help of (20) as follows :

Approximate ratio of any Bt o previous B	lies between*	Hence the exact value is
B_2 ...	0 and 1...	$1-\frac{1}{2}-\frac{1}{3} = \frac{1}{6}$
$B_4=\frac{3.4}{4\pi^2}B_2$	0 and 1...	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}-1 = \frac{1}{30}$
$B_6=\frac{5.6}{4\pi^2}B_4$	0 and 1...	$1-\frac{1}{2}-\frac{1}{3}-\frac{1}{7} = \frac{1}{42}$
$B_8=\frac{7.8}{4\pi^2}B_6$	0 and 1...	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}-1 = \frac{1}{30}$
$B_{10}=\frac{9.10}{4\pi^2}B_8$	0 and 1...	$1-\frac{1}{2}-\frac{1}{3}-\frac{1}{11} = \frac{5}{66}$
$B_{12}=\frac{11.12}{4\pi^2}B_{10}$	0 and 1...	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{15}-1 = \frac{691}{2730}$
$B_{14}=\frac{13.14}{4\pi^2}B_{12}$	0 and 2...	$2-\frac{1}{2}-\frac{1}{3} = \frac{7}{6}$
$B_{16}=\frac{15.16}{4\pi^2}B_{14}$	7 and 8 ..	$6+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{17} = \frac{3617}{510}$
$B_{18}=\frac{17.18}{4\pi^2}B_{16}$	54 and 55...	$56-\frac{1}{2}-\frac{1}{3}-\frac{1}{7}-\frac{1}{19} = \frac{43867}{798}$
$B_{20}=\frac{19.20}{4\pi^2}B_{18}$	529 and 530	$528+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{11} = \frac{174611}{330}$
... ...		

* These integral limits are got from a rough calculation of any B from the preceding B by the formula given in the first column.

$$\text{(ii)}\quad c_3\left(1-\frac{1}{2^{n-4}}\right)B_{n-3}-c_9\left(1-\frac{1}{2^{n-10}}\right)B_{n-9}+$$
$$c_{15}\left(1-\frac{1}{2^{n-16}}\right)B_{n-15}-\ldots$$
$$=\frac{2}{3}\cdot\frac{n}{2^n}\,3^{\frac{1}{2}(n-1)}+(-1)^{\frac{1}{2}(n-3)6}\quad\text{or}\quad(-1)^{\frac{1}{2}(n+1)6}$$
$$\text{or}\ (-1)^{\frac{1}{2}(n+5)6} \qquad \ldots(37)$$

17. The formulae obtained in §§ 1, 3, 4 may be called the one interval, two interval and three interval formulae respectively. The p interval formulae can be got by taking the pth roots of unity or of i according as p is odd or even.

For example, let us take the fifth roots of unity $(1,\alpha,\alpha^2,\alpha^3,\alpha^4)$, and find the 5 interval formulae.

Let $\phi(x)=\sin x+\sin x\alpha+\sin x\alpha^2+\sin x\alpha^3+\sin x\alpha^4$
$$=5\left(\frac{x^5}{5!}-\frac{x^{15}}{15!}+\frac{x^{25}}{25!}-\ldots\right).$$

Then it can easily be shewn that

$$16\sin x\sin x\alpha\sin x\alpha^2\sin x\alpha^3\sin x\alpha^4$$
$$=\phi(2x)-\phi\{2x(\alpha+\alpha^4)\}-\phi\{2x(\alpha^2+\alpha^3)\}$$
$$=\phi(2x)+\phi\{x(1+\sqrt{5})+\}\phi\{x(1-\sqrt{5})\,.\}$$

Taking logarithms and differentiating both sides, we have

$$5\left(B_0+B_{10}\frac{x^{10}}{10!}+B_{20}\frac{x^{20}}{20!}+B_{30}\frac{x^{30}}{30!}+\ldots\right)$$
$$=-x\,\frac{\frac{x^4}{4!}(1+\alpha_5)-\frac{x^{14}}{14!}(1+\alpha_{15})+\ldots}{\frac{x^5}{5!}(1+\alpha_5)-\frac{x^{15}}{15!}(1+\alpha_{15})+\ldots}, \qquad \ldots(38)$$

where $\alpha_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$,

so that $\alpha_n \alpha_m = \alpha_{n+m} + (-1)^n \alpha_{m-n}$.

(i) $$5\left(B_2 \frac{x^2}{2!} + B_{12} \frac{x^{12}}{12!} + B_{22} \frac{x^{22}}{22!} + \ldots\right)$$

$$= x \frac{\frac{x^6}{6!}(1+\alpha_7) - \frac{x^{16}}{16!}(1+\alpha_{17}) + \ldots}{\frac{x^5}{5!}(1+\alpha_5) - \frac{x^{15}}{15!}(1+\alpha_{15}) + \ldots}, \qquad \ldots(39$$

(ii) $$5\left(B_4 \frac{x^4}{4!} + B_{14} \frac{x^{14}}{14!} + B_{24} \frac{x^{24}}{24!} + \ldots\right)$$

$$= x \frac{\frac{x^8}{8!}(\alpha_7 - 1) - \frac{x^{18}}{18!}(\alpha_{17} - 1) + \ldots}{\frac{x^5}{5!}(1+\alpha_5) - \frac{x^{15}}{15!}(1+\alpha_{15}) + \ldots}, \qquad \ldots(40)$$

(iii) $$10\left(B_6 \frac{x^6}{6!} + B_{16} \frac{x^{16}}{16!} + B_{26} \frac{x^{26}}{26!} + \ldots\right)$$

$$= x \frac{\frac{x^{10}}{10!}(\alpha_{10} - 3) - \frac{x^{20}}{20!}(\alpha_{20} - 3) + \frac{x^{30}}{30!}(\alpha_{30} - 3) - \ldots}{\frac{x^5}{5!}(1+\alpha_5) - \frac{x^{15}}{15!}(1+\alpha_{15}) + \frac{x^{25}}{25!}(1+\alpha_{25}) - \ldots}, \qquad \ldots(41)$$

(iv) $$5\left(B_8 \frac{x^8}{8!} + B_{18} \frac{x^{18}}{18!} + B_{28} \frac{x^{28}}{28!} + \ldots\right)$$

$$= x \frac{\frac{x^{12}}{12!}(\alpha_{11} - 1) - \frac{x^{22}}{22!}(\alpha_{21} - 1) + \frac{x^{32}}{32!}(\alpha_{31} - 1 - \ldots)}{\frac{x^5}{5!}(1+\alpha_5) - \frac{x^{15}}{15!}(1+\alpha_{15}) + \frac{x^{25}}{25!}(1+\alpha_{25}) - \ldots},$$

$$\ldots(42)$$

Again from

$$16 \cos x \cos x\alpha \cos x\alpha^2 \cos x\alpha^3 \cos x\alpha^4$$
$$= 1 + \psi_1[2x] + \psi_1[2x(\alpha+\alpha^4)] + \psi_1[2x(\alpha^2+\alpha^3)],$$

where $\psi_1(x) = \cos x + \cos x\alpha + \cos x\alpha^2 + \cos x\alpha^3 + \cos x\alpha^4$

$$= 5\left(1 - \frac{x^{10}}{10!} + \frac{x^{20}}{20!} - \frac{x^{30}}{30!} + \ldots\right)$$

and similar relations, we can get many other identities.

18. The four interval formulae can be got from the following identities :

If $a_n = (1+\frac{1}{\sqrt{2}})^n + (1-\frac{1}{\sqrt{2}})^n$ and

$$b_n = (1+\frac{1}{\sqrt{2}})^n - (1-\frac{1}{\sqrt{2}})^n,$$

so that $a_{m+n} = a_m a_n - a_{m-n}/2^n$; and $b_{m+n} = a_m b_n + b_{m-n}/2^n$; then :

(i) $$4\left\{B_0 + B_8 \frac{x^8}{8!} + B_{16}\frac{x^{16}}{16!} + B_{24}\frac{x^{24}}{24!} + \ldots\right\}$$

$$= -x \frac{\frac{x^3}{3!}a_2 - \frac{x^{11}}{11!}a_6 + \frac{x^{19}}{19!}a_{10} - \ldots}{\frac{x^4}{4!}a_2 - \frac{x^{12}}{12!}a_6 + \frac{x^{20}}{20!}a_{10} - \ldots}, \qquad \ldots(43)$$

(ii) $$4\left\{B_2 \frac{x^2}{2!} + B_{10}\frac{x^{10}}{10!} + B_{18}\frac{x^{18}}{18!} + \ldots\right\}$$

$$= -x \frac{\frac{x^5}{5!}a_3 - \frac{x^{13}}{13!}a_7 + \frac{x^{21}}{21!}a_{11\ldots\infty}}{\frac{x^4}{4!}a_2 - \frac{x^{12}}{12!}a_6 + \frac{x^{20}}{20!}a_{10\ldots\infty}}, \qquad \ldots(44)$$

(iii) $$4\sqrt{2}\left\{B_4 \frac{x^4}{4!} + B_{12}\frac{x^{12}}{12!} + B_{20}\frac{x^{20}}{20!} + \ldots\right\}$$

where $\sigma_n = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \dots$.

Differentiating with respect to p, and equating the coefficients of p^{n-1}, we have

$$n\phi(n) = \sigma_1\phi(n-1) + \sigma_2\phi(n-2) + \sigma_3\phi(n-3) + \dots \text{to n terms.}$$

Thus we see that

$$\frac{\pi}{2}\phi(0) = 1 + \frac{1}{2}\frac{1}{3} + \frac{1.3}{2.4}\frac{1}{5} + \dots = \frac{\pi}{2},$$

$$\frac{\pi}{2}\phi(1) = 1 + \frac{1}{2}\frac{1}{3^2} + \frac{1.3}{2.4}\frac{1}{5^2} + \dots = \frac{\pi}{2}\log 2$$

$$\frac{\pi}{2}\phi(2) = 1 + \frac{1}{2}\frac{1}{3^3} + \frac{1.3}{2.4}\frac{1}{5^3} + \dots = \frac{\pi^7}{48} + \frac{\pi}{4}(\log 2)^2$$

$$\frac{\pi}{2}\phi(3) = 1 + \frac{1}{2}\frac{1}{3^4} + \frac{1.3}{2.4}\frac{1}{5^4} + \dots$$

$$= \frac{\pi^3}{48}\log 2 + \frac{\pi^3}{12}(\log 2)^3 + \frac{\pi}{6}\sigma_3 = \frac{\pi^3}{48}\log 2 + \frac{\pi}{12}(\log 2)^3 + \frac{\pi}{8}S_3,$$

and so on.

2. More generally, consider the series

$$\frac{1}{b^n} - \frac{a}{1!}\frac{1}{(b+1)^n} + \frac{a(a-1)}{2!}\frac{1}{(b+2)^n} - \dots$$

Writing $\frac{\Gamma(b)\Gamma(a+1)}{\Gamma(a+b+1)}\phi(n-1)$ for this, and taking the identity

$$\frac{1}{b+p} - \frac{a}{1!}\frac{1}{b+1+p} + \frac{a(a-1)}{2!}\frac{1}{b+2+p} - \dots$$

$$= \int_0^1 x^{b+p-1}(1-x)^a\,dx = \frac{\Gamma(b+p)\Gamma(a+1)}{\Gamma(a+b+p+1)},$$

we find

$$n\phi(n) = \sigma_1\phi(n-1) + \sigma_2\phi(n-2) + \sigma_3\phi(n-3) + \dots \text{to n terms.}$$

But $$\Gamma\left(\frac{p+1}{2}\right) = \frac{\pi^{\frac{1}{2}}\ \Gamma(p+1)}{2^p\ \Gamma\left(\frac{p+2}{2}\right)}$$

(vide Williamson, *Integral Calculus*, p. 164).

Therefore $$f(p) = \frac{\pi}{2^{p+1}\ \Gamma\left\{\frac{p+2}{2}\right\}^2} \cdot \Gamma(p+1)$$

Therefore log f(p)

$$= \log\ (\tfrac{1}{2}\pi) - p\ \log 2 + \frac{p^2}{2}\left(1 - \frac{1}{2}\right)S_2 - \frac{p^3}{3}\left(1 - \frac{1}{2^2}\right)S_3 + \ldots\ldots (1)$$

where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \ldots$ad inf. (vide Carr's *Synoposis* 2295).

Again, by expanding f (p) in ascending powers of p, we have

$$f(p) = \left(1 + \frac{1}{2}\ \frac{1}{3} + \frac{1.3}{2.4}\ \frac{1}{5} + \ldots\right) - p\left(1 + \frac{1}{2}\ \frac{1}{3^2} + \frac{1.3}{2.4}\ \frac{1}{5^2} + \ldots\right)$$

$$+ p^2\left(1 + \frac{1}{2}\ \frac{1}{3^3} + \frac{1.3}{2.4}\ \frac{1}{5^3} + \ldots\right) - \ldots$$

$$= \frac{\pi}{2}\{\phi(0) - p\phi(1) + p^2\phi(2) - p^3\phi(3) + \ldots,\}$$

where $\frac{1}{1^{n+1}} + \frac{1}{2} + \frac{1}{3^{n+1}} + \frac{1.3}{2.4}\ \frac{1}{5^{n+1}} + \ldots = \frac{\pi}{2}\phi(n)$.

Hence (1) may be written

$$\log \tfrac{1}{2}\pi + \log\{\phi(0) - p.\phi(1) + p^2.\phi(2) - \ldots\}$$

$$= \log\ (\tfrac{1}{2}\pi) - p\ \log 2 + \frac{p^2}{2}(1 - \tfrac{1}{2})S_2 - \frac{p^3}{3}\left(1 - \frac{1}{2^2}\right)S_3 + \ldots$$

$$= \log\ (\tfrac{1}{2}\pi) - p\sigma_1 + \frac{p^2}{2}\sigma_2 - \frac{p^3}{3}\sigma_3 + \ldots,$$

where $\sigma_n = \frac{1}{b^n} - \frac{1}{(a+b+1)^n} + \frac{1}{(b+1)^n} - \frac{1}{(a+b+2)^n} + \cdots$.

Examples : Put $a = -\frac{1}{2}$, $b = \frac{1}{4}$. Then we see that

(i) $1 + \frac{1}{2}\frac{1}{5} + \frac{1.3}{2.4}\frac{1}{9} + \frac{1.3.5}{2.4.6}\frac{1}{13} + \cdots = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}}$,

(ii) $1 + \frac{1}{2}\frac{1}{5^2} + \frac{1.3}{2.4}\frac{1}{9^2} + \frac{1.3.5}{2.4.6}\frac{1}{13^2} + \cdots = \frac{\{(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}}\frac{\pi}{4}$,

(iii) $1 + \frac{1}{2}\frac{1}{5^3} + \frac{1.3}{2.4}\frac{1}{9^3} + \frac{1.3.5}{2.4.6}\frac{1}{13^3} + \cdots = \frac{\{(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}}\left\{\frac{\pi^2}{32} + \frac{1}{2}S'_2, \right\}$

(iv) $1 + \frac{1}{2}\frac{1}{5^4} + \frac{1.3}{2.4}\frac{1}{9^4} + \frac{1.3.5}{2.4.6}\frac{1}{13^4} + \cdots$

$$= \frac{\{(\frac{1}{4})\}^2}{4\sqrt{(2\pi)}}\left\{\frac{5\pi^3}{384} + \frac{\pi}{8}S'_2 + \frac{1}{3}S'_3, \right\}$$

where $S'_r = \frac{1}{1^r} - \frac{1}{3^r} + \frac{1}{5^r} - \frac{1}{7^r} + \cdots$.

4

Note on a Set of Simultaneous Equations*

(*Journal of the Indian Mathematical Society, IV*, 1912, 94–96)

1. Consider the equations

$$x_1+x_2+x_3+\ldots+x_n=a_1,$$
$$x_1y_1+x_2y_2+x_3y_3+\ldots+x_ny_n=a_2,$$
$$x_1y_1^2+x_2y_2^2+x_3y_3^2+\ldots+x_ny_n^2=a_3,$$
$$x_1y_1^3+x_2y_2^3+x_3y_3^3+\ldots+x_ny_n^3=a_4,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_1y_1^{2n-1}+x_2y_2^{2n-1}+\ldots+x_ny_n^{2n-1}=a_{2n},$$

where $x_1,x_2,x_3,\ldots x_n$ and $y_1,y_2,y_3,\ldots y_n$ are 2n unknown quantities.

Now, let us take the expression

$$\phi(\theta)=\frac{x_1}{1-\theta y_1}+\frac{x_2}{1-\theta y_2}+\frac{x_3}{1-\theta y_3}+\ldots+\frac{x_n}{1-\theta y_n} \qquad \ldots(1)$$

and expand it in ascending powers of θ. Then we see that the expression is equal to

$$a_1+a_2\theta+a_3\theta^2+\ldots+a_{2n}\theta^{2n-1}+\ldots \qquad \ldots(2)$$

But (1), when simplified, will have for its numerator an expression of the (n–1) th degree in θ, and for its denominator an expression of the nth degree in θ.

*For a solution, by determinants, of a similar set of equations, see Burnside and Panton, *Theory of Equations*, Vol.II, p. 106, Ex.3 [Editor, J. Indian Math. Soc.]

where x,y,z,u,v,p,q,r,s,t are the unknowns. Proceeding as before, we have

$$\frac{x}{1-\theta p}+\frac{y}{1-\theta q}+\frac{z}{1-\theta r}+\frac{u}{1-\theta s}+\frac{v}{1-\theta t}=$$

$$2+3\theta+16\theta^2+31\theta^3+103\theta^4+235\theta^5+674\theta^6+1669\theta^7+4526\theta^8+11594\theta^9+\ldots$$

By the method of indeterminate coefficients, this can be shewn to be equal to

$$\frac{2+\theta+3\theta^2+2\theta^3+\theta^4}{1-\theta-5\theta^2+\theta^3+3\theta^4-\theta^5}.$$

Splitting this into partial fractions, we, get the values of the unknowns, as follows :

$$x=-\frac{3}{5}, \qquad p=-1,$$

$$y=\frac{18+\sqrt{5}}{10}, \qquad q=\frac{3+\sqrt{5}}{2},$$

$$z=\frac{18-\sqrt{5}}{10} \qquad r=\frac{3-\sqrt{5}}{2},$$

$$u=-\frac{8+\sqrt{5}}{2\sqrt{5}}, \qquad s=\frac{\sqrt{5}-1}{2},$$

$$v=\frac{8-\sqrt{5}}{2\sqrt{5}}, \qquad t=\frac{\sqrt{5}+1}{2}.$$

Thus we may suppose that

$$\phi(\theta)=\frac{A_1+A_2\theta+A_3\theta^2+\ldots+A_n\theta^{n-1}}{1+B_1\theta+B_2\theta^2+B_3\theta^3+\ldots+B_n\theta^n} \qquad \ldots(3)$$

$$=a_1+a_2\theta+a_3\theta^2+\ldots+a_{2n}\theta^{2n-1}+\ldots;$$

and so $(1+B_1\theta+\ldots)(a_1+a_2\theta+\ldots)=A_1+A_2\theta+\ldots$

Equating the coefficients of like powers of θ, we have

$$A_1=a_1,$$
$$A_2=a_2+a_1B_1,$$
$$A_3=a_3+a_2B_1+a_1B_2,$$
$$A_n=a_n+a_{n-1}B_1+a_{n-2}B_2+\ldots+a_1B_{n-1},$$
$$0=a_{n+1}+a_nB_1+\ldots+a_1B_n,$$
$$0=a_{n+2}+a_{n+1}B_1+\ldots\ldots a_2B_n,$$
$$0=a_{n+3}+a_{n+1}B_2+\ldots+a_3B_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$0=a_{2n}+a_{2n-1}B_1+\ldots+a_nB_n.$$

From these $B_1, B_2, \ldots B_n$ can easily be found, and since $A_1, A_2, \ldots A_n$ depend upon these values they can also be found.

Now, splitting (3) into partial fractions in the form

$$\frac{p_1}{1-q_1\theta}+\frac{p_2}{1-q_2\theta}+\frac{p_3}{1-q_3\theta}+\ldots+\frac{p_n}{1-q_n\theta},$$

and comparing with (1), we see that

$$x_1=p_1,\ y_1=q_1;$$
$$x_2=p_2,\ y_2=q_2;$$
$$x_3=p_3,\ y_3=q_3;$$
$$\ldots\ldots\ldots\ldots\ldots$$

2. As an example we may solve the equations:

$$x+y+z+u+v=2,$$
$$px+qy+rz+su+tv=3,$$
$$p^2x+q^2y+r^2z+s^2u+t^2v=16,$$
$$p^3x+q^3y+r^3z+s^3u+t^3v=31,$$
$$p^4x+q^4y+r^4z+s^4u+t^4v=103,$$
$$p^5x+q^5y+r^5z+s^5u+t^5v=235,$$
$$p^6x+q^6y+r^6z+s^6u+t^6v=674,$$
$$p^7x+q^7y+r^7z+s^7u+t^7v=1669,$$
$$p^8x+q^8y+r^8z+s^8u+t^8v=4526,$$
$$p^9x+q^9y+r^9z+s^9u+t^9v=11595,$$

5

Irregular Numbers

(*Journal of the Indian Mathematical Society*, V, 1913, 105–106)

1. Let $a_2, a_3, a_5, a_7, \ldots$ denote numbers less than unity, where the subscripts 2,3,5,7,...are the series of prime numbers. Then

$$\frac{1}{1-a_2}\cdot\frac{1}{1-a_3}\cdot\frac{1}{1-a_5}\cdots =$$

$1+a_2+a_3+a_2\,a_2+a_5+a_2.\,a_3+a_7+a_2.\,a_2.\,a_2+a_3.\,a_3+\ldots$, (1) the terms being so arranged that the products obtained by multiplying the subscripts are the series of natural numbers 2,3,4,5,6,7, 8,9,...

The above result is easily got if we remember that the natural numbers are formed by multiplying primes and their powers.

2. Similarly, we have

$$\frac{1}{1+a_2}\cdot\frac{1}{1+a_3}\cdot\frac{1}{1+a_5}\cdots$$

$= 1 - a_2 - a_3 + a_2.a_2 - a_5 + a_2.a_3 - a_7 - a_2.a_2.a_2 + a_3.a_3 + \ldots$,(2) where the sign is negative whenever a term contains an odd number of prime subscripts.

3. Put $a_2 = 1/2^n$, $a_3 = 1/3^n$, $a_5 = 1/5^n$,...in (1), and we get

$$\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\ldots = \frac{1}{S_n}, \qquad \ldots(3)$$

where S_n denotes $1/1^n+1/2^n+1/3^n+1/4^n+\ldots$

where 2,3,5,6,7,... are the numbers containing *dissimilar* prime divisors.

7. Also taking half the difference between § 3 and 4,

$$\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{11^n}+\frac{1}{13^n}+\frac{1}{17^n}+\frac{1}{19^n}+\frac{1}{23^n}+\frac{1}{29^n}$$

$$+\frac{1}{30^n}+\frac{1}{31^n}+\ldots=\frac{S^2_n-S_{2n}}{2S_nS_{2n}}, \quad \ldots(12)$$

where 2,3,5,...are numbers containing an *odd* number of *dissimilar* prime divisors,

Examples : (i) $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{9}{2\pi^2}$, ...(13)

(ii) $\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{15}{2\pi^4}$. ...(14)

8. Subtracting (11) from S_n, we have

$$\frac{1}{4^n}+\frac{1}{8^n}+\frac{1}{9^n}+\frac{1}{12^n}+\ldots=\frac{S_n(S_{2n}-1)}{S_{2n}}, \quad \ldots(14)$$

where 4,8,9,...are composite numbers having *at least* two *equal* prime divisors.

Changing n into 2n in (3) and dividing by the original, we obtain

$$(1+\frac{1}{2^n})\ (1+\frac{1}{3^n})\ (1+\frac{1}{5^n})\ (1+\frac{1}{7^n}).\ \ldots = \frac{S_n}{S_{2n}}. \qquad \ldots(4)$$

Examples : (i) $(1+\frac{1}{2^2})\ (1+\frac{1}{3^2})\ (1+\frac{1}{5^2})\ldots. = \frac{15}{\pi^2}$,....(5)

(ii) $(1+\frac{1}{2^4})\ (1+\frac{1}{3^4})\ (1+\frac{1}{5^4})\ldots. = \frac{105}{\pi^4}$,....(6)

since $S_2 = \pi^2/6,\ S_4 = \pi^4/90,\ S_8 = \pi^8/9450.$

4. Subtract (2) from (1) and put $a_2 = 2^{-n}\ldots$; then

$$\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{8^n}+\frac{1}{11^n}+\frac{1}{12^n}+\ldots = \frac{S^2_n - S_{2n}}{2S_n},$$

where the numbers 2,3,5,7,8,....contain an odd number of prime divisors.

Examples : (i) $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}+\ldots = \frac{\pi^2}{20}$, ...(7)

(ii) $\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{8^4}+\ldots = \frac{\pi^4}{1260}$ ....(8)

5. Again (2.3,5,7,...being the prime numbers)

$(1+a_2)\ (1+a_3)\ (1+a_5)\ (1+a_7)\ldots =$

$1+a_2+a_3+a_5+a_2.a_3+a_7+a_2.a_5+a_{11}+a_{13}+\ldots$, ...(9)

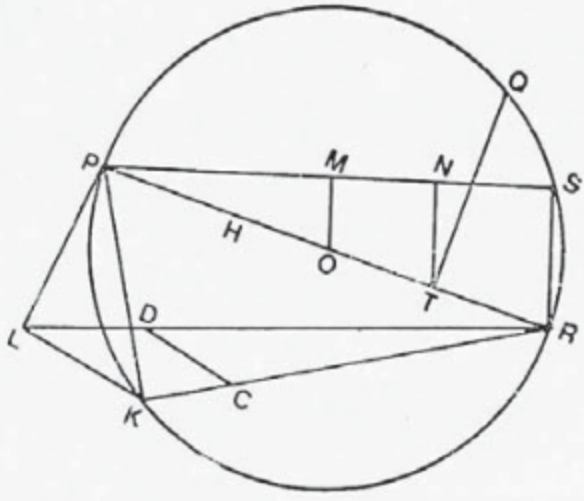
where the product of the subscripts in any term is a natural number containing *dissimilar* prime divisors; and

$(1-a_2)\ (1-a_3)\ (1-a_5)\ (1-a_7)\ldots = 1-a_2-a_3-a_5+a_2.a_3-a_7,\ldots.$ ...(10)

where the signs are negative whenever the number of factors is odd.

6. Replacing as before $a_2, a_3, a_5, \ldots$ by the values given in §3 and using (4), we deduce that

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\ldots = \frac{S_n}{S_{2n}}, \qquad \ldots(11)$$



Hence $RK^2 = PR^2 - PK^2 = \frac{113}{144} d^2$,

and $RL^2 = PR^2 + PL^2 = \frac{355}{324} d^2$.

But $\frac{RK}{RL} = \frac{PC}{RD} = \frac{3}{2}\sqrt{\frac{113}{355}}$,

and $RC = \frac{3}{4} d$.

Therefore $RD = \frac{d}{2}\sqrt{\frac{355}{113}} = r\sqrt{\pi}$ very nearly.

Note.—If the area of the circle be 140,000 square miles, then RD is greater than the true length by about an inch.

6

Squaring the Circle

(*Journal of the Indian Mathematical Society*, V, 1913,132)

Let PQR be a cricle with centre O, of which a diameter is PR. Bisect PO at H and let T be the point of trisection of OR nearer R. Draw TQ perpendicular to PR and place the chord RS = TQ.

Join PS, and draw OM and TN parallel to RS. Place a chord PK = PM, and draw the tangent PL = MN. Join RL, RK and KL. Cut off RC = RH. Draw CD parallel to KL, meeting RL, at D.

Then the square on RD will be equal to the circle PQR approximately. For

$$RS^2 = \frac{5}{36} d^2 ,$$

where d is the diameter of the circle.

Therefore $PS^2 = \frac{31}{36} d^2$,

But PL and PK are equal to MN and PM respectively.

Therefore $PK^2 = \frac{31}{144} d^2$, and $PL^2 = \frac{31}{324} d^2$.

7

On the Integral

$$\int_0^x \frac{\tan^{-1}t}{t}dt$$

(*Journal of the Indian Mathematical Society*, VII, 1915, 93-96)

1. Let $$\phi(x)=\int_0^x \frac{\tan^{-1}t}{t}dt. \qquad ...(1)$$

Then it is easy to see that

$$\phi(x)+\phi(-x)=0; \qquad ...(2)$$

and that $$\phi(x)=\frac{x}{1^2}-\frac{x^3}{3^2}+\frac{x^5}{5^2}-\frac{x^7}{7^2}+\cdots \qquad ...(3)$$

provided that $|x| \leqslant 1$.

Changing t into 1/t in (1), we obtain

$$\phi(x)-\phi\left(\frac{1}{x}\right)=\frac{1}{2}\pi \log x, \qquad ...(4)$$

provided that the real part of x is positive.

The results in the following two sections can be very easily proved by differentiating both sides with respect to x.

2. If $0<x<\frac{1}{2}\pi$, than

$$\frac{\sin 2x}{1^2}+\frac{\sin 6x}{3^2}+\frac{\sin 10x}{5^2}+\ldots=\phi(\tan x)-x\log(\tan x). \qquad ...(5)$$

If, in particular, we put $x=\frac{1}{8}\pi$ and $\frac{1}{12}\pi$ in (5), we obtain.

$$\frac{1}{1^2}-\frac{1}{5^2}+\frac{1}{9^2}-\frac{1}{13^2}+\ldots$$

$$=\frac{1}{\sqrt{2}}\phi(\sqrt{2}-1)+\frac{\pi}{8\sqrt{2}}\log(1+\sqrt{2})+\frac{\pi^2}{16}; \qquad \ldots(6)^*$$

and $$2\phi(1)=3\phi(2-\sqrt{3})+\frac{1}{4}\pi\log(2+\sqrt{3}). \qquad \ldots(7)$$

If $-\frac{1}{2}\pi<x<\frac{1}{2}\pi$, then

$$2\phi(\tan\frac{x}{2})=\sin x+\frac{2}{3}\frac{\sin^2 x}{3})+\frac{2.4\sin^5 x}{3.5\ 5}+\ldots \qquad \ldots(8)$$

If $0<x<\frac{1}{2}\pi$, then

$$\frac{\sin x}{1^2}\cos x+\frac{\sin 2x}{2^2}\cos^2 x+\frac{\sin 3x}{3^2}\cos^3 x+\ldots$$

$$=\phi(\tan x)+\tfrac{1}{2}\pi\log\cos x-x\log\sin x; \qquad \ldots(9)$$

and $$\frac{\cos x+\sin x}{1^2}+\frac{1}{2}\frac{\cos^3 x+\sin^3 x}{3^2}+\frac{1.3}{2.4}\frac{\cos^5 x+\sin^5 x}{5^2}+\ldots$$

$$=\phi(\tan x)+\tfrac{1}{2}\pi\log(2\cos x). \qquad \ldots(10)$$

If $-\frac{1}{2}\pi<x<\frac{1}{2}\pi$ and a be any number such that

$$|(1-a)\sin x|<1,\ |(1-\frac{1}{a})\cos x|<1,$$

then $$\frac{\sin x}{1^2}(1-\frac{1}{a})\cos x+\frac{\sin 2x}{2^2}(1-\frac{1}{a})^2\cos^2 x+$$

$$\frac{\sin 3x}{3^2}(1-\frac{1}{a})^3\cos^3 x+\frac{\sin(x+\frac{1}{2}\pi)}{1^2}(1-a)\sin x$$

$$-\frac{\sin 2(x+\frac{1}{2}\pi)}{2^2}(1-a)^2\sin^2 x+\ldots=\phi(\tan x)-\phi(a\tan x)+x\log a. \qquad \ldots(11)$$

---

* The equation should read

$$\frac{1}{1^2}+\frac{1}{3^2}-\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}\ldots=\sqrt{2}\,\phi(\sqrt{2}-1)+4\,\frac{\pi}{\sqrt{2}}\log(1+\sqrt{2})$$

3. Let R(x) and I(x) denote the real and the imaginary parts of x respectively. Then, if $-1<R(x)<1$,

$$\log\left(1-\frac{x^2}{1^2}\right)-3\log\left(1-\frac{x^2}{3^2}\right)+5\log\left(1-\frac{x^2}{5^2}\right)-\ldots$$

$$=\frac{4}{\pi}\left[\phi(1)-\phi\left\{\tan\frac{1}{4}\pi(1-x)\right\}\right]+\log\tan\frac{1}{4}\pi(1-x). \quad \ldots 12)$$

Putting $x=\frac{2}{3}$ in (12) and using (7), we obtain

$$\left(1-\frac{4}{3^2}\right)\left(1-\frac{4}{9^2}\right)^{-3}\left(1-\frac{4}{15^2}\right)\cdots\left(1-\frac{4}{21^2}\right)^{-7}\left(1-\frac{4}{27^2}\right)^{9}\cdots$$

$$=(2-\sqrt{3})^{\frac{2}{3}}\,e^{n}, \text{ where } n=\frac{4}{3\pi}\phi(1). \quad \ldots (13)$$

Again, subtracting $\log(1-x)$ from both sides in (12) and making $x\to 1$, we obtain

$$\left(1-\frac{1}{3^2}\right)^{-3}\left(1-\frac{1}{5^2}\right)^{5}\left(1-\frac{1}{7^2}\right)^{-7}\left(1-\frac{1}{9^2}\right)^{9}\cdots=\frac{\pi}{8}e^{3n}. \quad \ldots (14)$$

If $-1<I(x)<1$, then

$$\log\left(1+\frac{x^2}{1^2}\right)-3\log\left(1+\frac{x^2}{3^2}\right)+5\log\left(1+\frac{x^2}{5^2}\right)-\ldots$$

$$=\frac{4}{\pi}\left[\phi(1)-\phi\left(e^{-\frac{1}{2}\pi x}\right)\right]-2x\tan^{-1}e^{-\frac{1}{2}\pi x}. \quad \ldots(15)$$

From this and (7) we see that, if $\frac{1}{2}\pi x=\log(2+\sqrt{3})$, then

$$\left(1+\frac{x^2}{1^2}\right)\left(1+\frac{x^2}{3^2}\right)^{-3}\left(1+\frac{x^2}{5^2}\right)^{5}\left(1+\frac{x^2}{5^2}\right)^{-7}\cdots=e^{n}, \quad \ldots(16)$$

where n is the same as in (13).

It follows at once from (12) and (15) that, if

$$-1<R(\beta)<1,\ -1<I(\alpha)<1,$$

then $$e^{\frac{1}{2}\pi\alpha\beta}=\left[\frac{1^2+\alpha^2}{1^2-\beta^2}\right]\left[\frac{3^2-\beta^2}{3^2+\alpha^2}\right]^{3}\left[\frac{5^2+\alpha^2}{5^2-\beta^2}\right]^{5}\left[\frac{7^2-\beta^2}{7^2+\alpha^2}\right]^{7}\cdots$$

$\ldots(17)$

and, if n is any even integer, then

$$(1+\frac{4n^4}{1^4})\ (1+\frac{4n^4}{3^4})^{-3}(1+\frac{4n^4}{5^4})^{5}\ (1+\frac{4n^4}{7^4})^{-7}\ \ldots$$

$$= \exp\left\{\frac{8}{\pi}\ \phi(1)-\frac{8}{\pi}(-1)^{\frac{1}{2}n}\left[\phi(e^{-\frac{1}{2}\pi n})+\tfrac{1}{2}\pi n\tan^{-1}e^{-\frac{1}{2}\pi n}\right]\right\}. \qquad \ldots(21)$$

Similarly from (19) we see that, if n is any positive odd integer, then

$$\tan^{-1}\frac{2n^2}{1^2}-3\tan^{-1}\frac{2n^2}{3^2}+5\tan^{-1}\frac{2n^2}{5^2}-\ldots$$

$$=\frac{4}{\pi}(-1)^{\frac{1}{2}(n-1)}\left\{\frac{\pi n}{4}\log\left(\frac{1+e^{-\frac{1}{2}\pi n}}{1-e^{-\frac{1}{2}\pi n}}\right)+\frac{1}{1^2}e^{-\frac{1}{2}\pi n}+\right.$$

$$\left.-\frac{1}{5^2}e^{-\frac{5}{2}\pi n}+\ldots\right\}; \qquad (22)$$

and, if n is a positive even integer, then

$$\tan^{-1}\frac{2n^2}{1^2}-3\tan^{-1}\frac{2n^2}{3^2}+5\tan^{-1}\frac{2n^2}{5^2}-\ldots$$

$$=2n(-1)^{\frac{1}{2}n-1}\tan^{-1}e^{-\frac{1}{2}\pi n}. \qquad \ldots(23)$$

In this connection it may be interesting to note that

$$\tan^{-1}e^{-\frac{1}{2}\pi n}=\frac{\pi}{4}-\left[\tan^{-1}\frac{n}{1}-\tan^{-1}\frac{n}{3}+\tan^{-1}\frac{n}{5}-\ldots\right] \ldots(24)$$

for all real values of n.

5. Remembering that

$$\frac{\pi}{4\cosh\pi x}=\frac{1}{1^2+4x^2}-\frac{3}{3^2+4x^2}+\frac{5}{5^2+4x^2}-\ldots$$

provided that $\cosh \frac{1}{2}\pi\alpha = \sec \frac{1}{2}\pi\beta$.

4. Now changing x into $2x(1+i)$ in (15), we have

$$\log\left(1+\frac{8ix^2}{1^2}\right)-3\log\left(1+\frac{8ix^2}{3^2}\right)+5\log\left(1+\frac{8ix^2}{5^2}\right)-\ldots$$

$$=\frac{4}{\pi}\phi(1)-4x(1+i)\tan^{-1}e^{-\pi x(1+i)}-\frac{4}{\pi}\left\{\frac{1}{1^2}e^{-\pi x(1+i)}\right.$$

$$\left.-\frac{1}{3^2}e^{-3\pi x(1+i)}\right\}+\ldots.$$

Equating real and imaginary parts we see that, if x is positive, then

$$\log\left(1+\frac{64x^4}{1^4}\right)-3\log\left(1+\frac{64x^4}{3^4}\right)+5\log\left(1+\frac{64x^4}{5^4}\right)-\ldots$$

$$=\frac{8}{\pi}\phi(1)-2x\log\left(\frac{\cosh\pi x+\sin\pi x}{\cosh\pi x-\sin\pi x}\right)-4x\tan^{-1}\left(\frac{\cos\pi x}{\sinh\pi x}\right)$$

$$-\frac{8}{\pi}\left\{\frac{\cos\pi x}{1^2}e^{-\pi x}-\frac{\cos 3\pi x}{3^2}e^{-3\pi x}+\frac{\cos 5\pi x}{5^2}e^{-5\pi x}-\ldots\right\}\ldots(18)$$

$$\text{and } \tan^{-1}\frac{8x^2}{1^2}-3\tan^{-1}\frac{8x^2}{3^2}+5\tan^{-1}\frac{8x^2}{5^2}-\ldots$$

$$=x\log\left(\frac{\cosh\pi x+\sin\pi x}{\cosh\pi x-\sin\pi x}\right)-2x\tan^{-1}\left(\frac{\cos\pi x}{\sinh\pi x}\right)$$

$$+\frac{4}{\pi}\left\{\frac{\sin\pi x}{1^2}e^{-\pi x}-\frac{\sin 3\pi x}{3^2}e^{-3\pi x}+\frac{\sin 5\pi x}{5^2}e^{-5\pi x}-\ldots\right\}\ldots.(19)$$

It follows from (18) that, if n is a positive odd integer, then

$$\left(1+\frac{4n^4}{1^4}\right)\left(1+\frac{4n^4}{3^4}\right)^{-3}\left(1+\frac{4n^4}{5^4}\right)^{5}\left(1+\frac{4n^4}{7^4}\right)^{-7}\ldots$$

$$e^{\frac{8}{\pi}\phi(1)}\left[\frac{1-e^{-\frac{1}{2}\pi n}}{1+e^{-\frac{1}{2}\pi n}}\right]^{2n\cos\frac{1}{2}(n-1)\pi} \qquad \ldots.(20)$$

we have

$$\frac{\pi}{4}\sum_{1}^{\infty}\frac{1}{n^2\cosh \pi n x}=\sum_{n=1}^{n=\infty}\left\{\frac{1}{n^2(1^2+4n^2x^2)}-\frac{3}{n^2(3^2+4\pi n^2x^2)}+\cdots\right\}$$

$$=\frac{\pi^3}{8}\left(\frac{1}{3}+\frac{x^2}{2}\right)-\pi x\left[\frac{\coth\frac{\pi}{2x}}{1^2}-\frac{\coth\frac{3\pi}{2x}}{3^2}+\frac{\coth\frac{5\pi}{2x}}{5^2}-\cdots\right] \quad \ldots(25)$$

That is to say, if α and β are real and $\alpha\beta=\pi^2$, then

$$\phi(1)+2\phi(e^{-\alpha})+2\phi(e^{-2\alpha})+2\phi(e^{-3\alpha})+\cdots$$

$$=\frac{\pi}{8}\left(\frac{\alpha}{3}+\frac{\beta}{2}\right)-\frac{\pi}{4\beta}\left\{\frac{1}{1^2\cosh\beta}+\frac{1}{2^2\cosh 2\beta}+\cdots\right\} \quad \ldots(26)$$

If. in particular, we put $\alpha=\beta=\pi$ in (26), we obtain

$$\phi(1)=\frac{5\pi^2}{48}-2\left\{\frac{1}{1^2(e^{\pi}-1)}-\frac{1}{3^2(e^{3\pi}-1)}+\frac{1}{5^2(e^{5\pi}-1)}\cdots\right\}$$

$$-\tfrac{1}{2}\left\{\frac{1}{1^2(e^{\pi}+e^{-\pi})}+\frac{1}{2^2\pi(e^{2}+e^{-2\pi})}+\frac{1}{3^2(e^{3\pi}+e^{-3\pi})}+\cdots\right\}$$

$$=.9159655942, \quad \ldots(27)$$

approximately.

But $\frac{1}{n}\{(1+a_1)\log p_1+(1+a_2)\log p_2+\ldots+(1+a_n)\log p_n\}$

$$> \{(1+a_1)(1+a_2)\ldots(1+a_n)\log p_1 \log p_2 \ldots \log p_n\}^{\frac{1}{n}}, \quad \ldots(4)$$

since the arithmetic mean of unequal positive numbers is always greater than their geometric mean. Hence $\frac{1}{n}\{\log p_1+\log p_2+\ldots+\log p_n+\log N\} > \left\{\log p_1 \log p_2 \ldots \log p_n . d(N)\right\}^{\frac{1}{n}}$

In other words

$$d(N) < \frac{\{\frac{1}{n}\log (p_1p_2p_3\ldots p_nN)\}^n}{\log p_1 \log p_2 \log p_3 \ldots \log p_n} \quad \ldots(5)$$

for all values of N whose prime divisors are $p_1, p_2, p_3, \ldots p_n$.

3. Next let us consider the case in which only the number of prime divisors of N is known. Let

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_n^{a_n},$$

where n is a given number; and let

$$N' = 2^{a_1} .3^{a_2} ,5^{a_3} \ldots p^{a_n},$$

where p is the natural nth prime. Then it is evident that

$$N' \leqslant N; \quad \ldots(6)$$

and

$$d(N') = d(N). \quad \ldots(7)$$

But

$$d(N') < \frac{\{\frac{1}{n}\log (2.3.5\ldots p, N')\}^n}{\log 2 \log 3 \log 5 \ldots \log p}, \quad \ldots(8)$$

by virtue of (5). It follows from (6) to (8) that, if p be the natural nth prime, then

$$d(N) < \frac{\{\frac{1}{n}\log (2.3.5\ldots p.N)\}^n}{\log 2 \log 3 \log 5 \ldots \log p}, \quad \ldots(9)$$

for all values of N having n prime divisors.

8

On the Number of Divisors of a Number

(*Journal of the Indian Mathematical Society*, VII, 1915.131–133)

1. If δ be divisor of N, then there is a conjugate divisor δ' such that $\delta\delta' = N$. Thus we see that

the number of divisors from 1 to $\sqrt{N}$ is equal to the number of divisors from $\sqrt{N}$ to N. ...(1)

From this it evidently follows that

$$d(N) < 2\sqrt{N}, \qquad \text{....(2)}$$

where d(N) denotes the number of divisors of N (including unity and the number itself). This is only a trivial result, as all the numbers from 1 to $\sqrt{N}$ cannot be divisors of N. So let us try to find the best possible superior limit for d(N) by using purely elementary reasoning.

2. First let us consider the case in which all the prime divisors of N are known. Let

$$N = p_1^{a_1} p_2^{a_2} \cdot p_3^{a_3} \cdots p_n^{a_n},$$

where $p_1, p_2, p_3 \ldots p_n$ are a given set of n primes. Then it is easy to see that

$$d(N) = (1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n). \qquad \text{...(3)}$$

4. Finally, let us consider the case in which nothing is known about N. Any integer N can be written in the form

$$2^{a_2}\,3^{a_3}\,5^{a_5}\ldots,$$

where $a > 0$. Now let

$$x^h = 2, \qquad \ldots(10)$$

where h is any positive number. Then we have

$$\frac{d(N)}{N^h} = \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{3^{ha_3}} \cdots \frac{1+a_5}{5^{ha_5}} \cdots \qquad \ldots(11)$$

But from (10) we see that, if q be any prime greater than x, then

$$\frac{1+a_q}{q^{ha_q}} \leqslant \frac{1+a_q}{x^{ha_q}} = \frac{1+a_q}{2^{a_q}} < 1. \qquad \ldots(12)$$

It follows from (11) and (12) that, if p be the largest prime not exceeding x, then

$$\frac{d(N)}{N^h} \leqslant \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{3^{ha_3}} \cdot \frac{1+a_5}{5^{ha_5}} \cdots \frac{1+a_p}{p^{ha_p}}$$

$$\leqslant \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{2^{ha_3}} \cdot \frac{1+a_5}{2^{ha_5}} \cdots \frac{1+a_p}{2^{ha_p}}. \qquad \ldots(13)$$

But it is easy to shew that the maximum value of $(1+a)2^{-ha}$ for the vnriable a is $\frac{2^h}{h\ e \log 2}$. Hence

$$\frac{d(N)}{N^h} \leqslant \left(\frac{2^h}{h\ e \log 2}\right)^{\omega(x)}, \qquad \ldots(14)$$

where $\omega(x)$ denotes the number of primes not exceeding x. But from (10) we have

$$h = \frac{\log 2}{\log x}.$$

Substituting this in (14), we obtain

$$d(N) \leqslant N^{\frac{\log 2}{\log x}} \left\{ \frac{\log 2}{\frac{2^{\log x}}{e\,(\log 2)^2} \log x} \right\}^{\omega(x)}. \qquad \ldots(15)$$

But it is easy to verify that, if $x \geqslant 6.05$, then

$$2^h < e(\log 2)^2.$$

From this and (15) it follows that, if $x \geqslant 6.05$, then

$$d(N) < 2^{(\log N)/(\log x)} (\log x)^{\omega(x)} \quad \ldots(16)$$

for all values of N, $\omega(x)$ being the number of primes not exceeding x.

5. The symbol "O" is used in the following sense :

$$\phi(x) = O\{\psi(x)\}$$

means that there is a positive constant K such that—

$$\left|\frac{\phi(x)}{\psi(x)}\right| \leqslant K$$

for all sufficiently large values of x (see Hardy, *Orders of Infinity*. pp. 5 et seq.) For example;

5(x) $O(x); \frac{1}{2}x = O(x)$; $x \sin x = O(x)$; $\sqrt{x} = O(x)$; $\log x = O(x)$; but $x^2 \neq O(x)$; $x \log x \neq O(x)$.

Hence it is obvious that

$$\omega(x) = O(x). \quad \ldots(17)$$

Now, let us suppose that

$$x = \frac{\log N}{(\log \log N)^2}$$

in (16). Then we have

$$\log x = \log \log N + O(\log \log \log N);$$

and so $$\frac{\log N}{\log x} = \frac{\log N}{\log \log N} + O\left(\frac{\log N \log \log \log N}{(\log \log N)^2}\right). \quad \ldots(18)$$

Again

$$\omega(x) \log \log x = 0 (x \log \log x) = O\left(\frac{\log N \log \log \log N}{(\log \log N)^2}\right) \quad \ldots(19)$$

It follows from (16), (18) and (19), that

$$\log d(N) < \frac{\log 2 \log N}{\log \log N} + O\left(\frac{\log N \log \log \log N}{(\log \log N)^2}\right) \quad \ldots(20)$$

for all sufficiently large values of N.

9

On the Sum of the Square Roots of the First n Natural Numbers

(*Journal of the Indian Mathematical Society*, VII, 1915, 173-175)

1. Let $\phi_1(n)=\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n}-(C_1+\frac{2}{3}n\sqrt{n}+\frac{1}{2}\sqrt{n})$,

$$-\frac{1}{6}\sum_{\nu=0}^{\nu=\infty}\{\sqrt{(n+\nu)}+\sqrt{(n+\nu+1)}\}^{-3},$$

where C_1 is a constant such that $\phi_1(1)=0$. Then we see that

$$\phi_1(n)-\phi_1(n+1)=-\sqrt{(n+1)}+[\frac{2}{3}(n+1)\sqrt{(n+1)}+\frac{1}{2}\sqrt{(n+1)}]$$

$$-(\frac{2}{3}n\sqrt{n}+\frac{1}{2}\sqrt{n})+\frac{1}{6}\{\sqrt{n}-\sqrt{(n+1)}\}^3=0.$$

But $\phi_1(1)=0$. Hence $\phi_1(n)=0$ for all values of n. That is to say

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\ldots+\sqrt{n}=C_1+\frac{2}{3}n\sqrt{n}+\frac{1}{2}\sqrt{n}$$

$$+\frac{1}{6}[\{\sqrt{n}+\sqrt{(n+1)}\}^{-3}+\{\sqrt{(n+1)}+\sqrt{(n+2)}\}^{-3}+$$

$$\{\sqrt{(n+2)}+\sqrt{(n+3)}\}^{-3}+\ldots].$$

But it is known that

$$C_1 = -\frac{1}{4\pi}\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \ldots\right) \tag{2}$$

Putting $n = 1$ in (1) and using (2), we obtain

$$2\pi\left\{\frac{1}{(\sqrt{1})^3} + \frac{1}{(\sqrt{1}+\sqrt{2})^3} + \frac{1}{(\sqrt{2}+\sqrt{3})^3} + \frac{1}{(\sqrt{3}+\sqrt{4})^3} + \ldots\right\}$$

$$= 3\left\{\frac{1}{(\sqrt{1})^3} + \frac{1}{(\sqrt{2})^3} + \frac{1}{(\sqrt{3})^3} + \frac{1}{(\sqrt{4})^3} + \ldots\right\} \tag{3}$$

2. Again let

$$\phi_2(n) = 1\sqrt{1} + 2\sqrt{2} \ldots + n\sqrt{n} - \left(C_2 + \frac{2}{5}n^2\sqrt{n} + \tfrac{1}{2}n\sqrt{n} + \frac{1}{8}\sqrt{n}\right)$$

$$- \frac{1}{40}\sum_{\nu=0}^{\nu=\infty}\left[\sqrt{(n+\nu)} + \sqrt{(n+\nu+1)}\right]^{-5},$$

where C_2 is a constant such that $\phi_2(1) = 0$. Then we have

$$\phi_2(n) - \phi_2(n+1) = -(n+1)\sqrt{(n+1)}$$

$$+ \left\{\frac{2}{5}(n+1)^2\sqrt{(n+1)} + \tfrac{1}{2}(n+1)\sqrt{(n+1)} + \frac{1}{8}\sqrt{(n+1)}\right\}$$

$$- \left\{\frac{2}{5}n^2\sqrt{n} + \tfrac{1}{2}n\sqrt{n} + \frac{1}{8}\sqrt{n}\right\} + \frac{1}{40}\left\{n - \sqrt{(n+1)}\right\}^5 = 0.$$

But $\phi_2(1) = 0$. Hence $\phi_2(n) = 0$. In other words

$$1\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \ldots n\sqrt{n} = C_2 + \frac{2}{5}n^2\sqrt{n} + \tfrac{1}{2}n\sqrt{n} + \frac{1}{8}\sqrt{n}$$

$$+ \frac{1}{40}\left[\left(\sqrt{n} + \sqrt{(n+1)}\right)^{-5} + \left(\sqrt{(n+1)} + \sqrt{(n+2)}\right)^{-5} + \right.$$

$$\left. \left(\sqrt{(n+2)} + \sqrt{(n+3)}\right)^{-5} + \ldots\right] \tag{4}$$

But it is known that

$$C_2 = -\frac{3}{16\pi^2}\left(\frac{1}{1^2\sqrt{1}} + \frac{1}{2^2\sqrt{2}} + \frac{1}{3^2\sqrt{3}} + \ldots\right) \tag{5}$$

It is easy to see from (4) and (5) that

$$2\pi^2\left\{\frac{1}{(\sqrt{1})^5}+\frac{1}{(\sqrt{1}+\sqrt{2})^5}+\frac{1}{(\sqrt{2}+\sqrt{3})^5}+\frac{1}{(\sqrt{3}+\sqrt{4})^5}+\ldots.\right\}$$

$$=15\left\{\frac{1}{(\sqrt{1})^5}+\frac{1}{(\sqrt{2})^5}+\frac{1}{(\sqrt{3})^5}+\frac{1}{(\sqrt{4})^5}+\ldots.\right\} \qquad (6)$$

3. The corresponding results for higher powers are not so neat as the previous ones. Thus for example

$$1^2\sqrt{1}+2^2\sqrt{2}+3^2\sqrt{3}+\ldots+n^2\sqrt{n}$$

$$=C_3+\sqrt{n}\left(\frac{2}{7}n^3+\frac{1}{2}n^2+\frac{5}{24}n\right)$$

$$-\frac{1}{9}[(\sqrt{n}+\sqrt{(n+1)})^{-3}+(\sqrt{(n+1)}+\sqrt{(n2+)})^{-3}+\ldots.]$$

$$+\frac{1}{224}[(\sqrt{n}+\sqrt{(n+1)})^{-7}+(\sqrt{(n+1)}+(n+2))^{-7}$$

$$+\{\sqrt{(n+2)}+\sqrt{(n+3)}\}^{-7}+\ldots]; \qquad \ldots(7)$$

$$1^3\sqrt{1}+2^3\sqrt{2}+\ldots+n^3\sqrt{n}=C_4+\sqrt{n}\left(\frac{9}{6}n^4+\frac{1}{2}n^3+\frac{7}{24}n^2-\frac{7}{384}\right)$$

$$-\frac{1}{192}[\{\sqrt{n}+\sqrt{(n+1)}\}^{-5}+\{\sqrt{(n+1)}+\sqrt{(n+2)}\}^{-5}+\ldots.]$$

$$+\frac{1}{1152}[\{\sqrt{n}+\sqrt{(n+1)}\}^{-9}+\{\sqrt{(n+1)}+\sqrt{(n+2)}\}^{-9}+\ldots.]; \qquad \ldots(8)$$

and so on.

The constants C_3, C_4,...can be ascertained from the well-known result that the constant in the approximate summation of the series $1^{r-1}+2^{r-1}+3^{r-1}+\ldots+n^{r-1}$ is

$$\frac{2\Gamma(r)}{(2\pi)^r}\left(\frac{1}{1^r}+\frac{1}{2^r}+\frac{1}{3^r}+\frac{1}{4^r}+\ldots\right)\cos\frac{1}{2}\pi r, \qquad \ldots(9)$$

provided that real part of r is greater than 1.

4. Similarly we can shew, by induction, that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots.+\frac{1}{\sqrt{n}}=C_0+2\sqrt{n}+\frac{1}{2\sqrt{n}}$$

$$-\frac{1}{2}\left[\frac{\{\sqrt{n}+\sqrt{(n+1)}\}^{-3}}{\sqrt{\{n(n+1)\}}}+\frac{\{\sqrt{(n+1)}+\sqrt{(n+2)}\}^{-3}}{\sqrt{\{(n+1)\ (n+2)\}}}+\ldots\right] \ldots(10)$$

The value of C_0 can be determined as follows: from (10) we have

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots.+\frac{1}{\sqrt{(2n)}}-2\sqrt{(2n)}\to C_0, \quad \ldots(11)$$

as $n\to\infty$. Also

$$2\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{6}}+\ldots.+\frac{1}{\sqrt{(2n)}}\right)-2\sqrt{(2n)}\to C_0\sqrt{2}, \quad \ldots(12)$$

as $n\to\infty$.

Now subtracting (12) From (11) we see that

$$\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots.-\frac{1}{\sqrt{(2n)}}\to C_0(1-\sqrt{2}),$$

as $n\to\infty$.

That is to say

$$C_0=-(1+\sqrt{2})\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots.\right), \quad \ldots(13)$$

We can also show, by induction, that

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n}=C_1+\frac{2}{3}n\sqrt{n}+\frac{1}{2}\sqrt{n}+\frac{1}{24\sqrt{n}}$$

$$-\frac{1}{24}\left[\frac{\{\sqrt{n}+\sqrt{(n+1)}\}^{-5}}{\sqrt{\{n(n+1)\}}}+\frac{\{\sqrt{(n+1)}+\sqrt{(n+2)}\}^{-5}}{\sqrt{\{(n+1)\,(n+2)\}}}+\ldots,\right]. \quad \ldots(14)$$

The asymptotic expansion of $\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n}$ for large values of n can be shown to be

$$C_1+\frac{2}{3}\,n\sqrt{n}+\frac{1}{2}\sqrt{n}+\frac{1}{\sqrt{n}}\left(\frac{1}{24}-\frac{1}{1920n^2}+\frac{1}{9216n^4}-\ldots\right),\ldots(15)$$

by using the Euler-Maclaurin sum formula.

10

On the Product

$$\prod_{n=0}^{\infty} \left[1+\left(\frac{x}{a+nd}\right)^3 \right]$$

(*Journal of the Indian Mathematical Society*, VII. 1915, 209-211)

1. Let $\phi(\alpha,\beta) = \left\{1+\left(\frac{\alpha+\beta}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{2+\alpha}\right)^3\right\}$ ... ...(1)

It is easy to see that

$$\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{n+\beta}\right)^3\right\}$$

$$= \frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{\beta+2\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3} \left[1-\left\{\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]$$

$$\times\left[1-\left\{\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]; \qquad \ldots(2)$$

$$\prod_{n=1}^{n=\infty}\left\{\frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{\beta+2\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}\right\} =$$

$$\frac{\{\Gamma(1+\alpha)\,\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)}, \qquad \ldots(3)$$

$$\prod_{n=1}^{n=\infty}\left[1-\left\{\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]\left[1-\left\{\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}\right]^2$$

$$=\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha+\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)} \quad \ldots(4)$$

It follows from (1)-(4) that

$$\phi(\alpha,\beta)\,\phi(\beta,\alpha)$$

$$=\frac{\{\Gamma(1+\alpha)\,\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\,\Gamma(1+\beta+2\alpha)}\left\{\frac{\cosh(\alpha+\beta)\sqrt{3}-\cos(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}\right\} \quad (5)$$

But it is evident that, if $\alpha-\beta$ be any integer, then $\phi(\alpha,\beta)/\phi(\beta,\alpha)$ can be expressed in finite terms. From this and (5) it follows that $\phi(\alpha,\beta)$ can be expressed in finite terms, if $\alpha-\beta$ be any integer. That is to say

$$\left\{1+\left(\frac{x}{a}\right)^3\right\}\left\{1+\left(\frac{x}{a+d}\right)^3\right\}\left\{1+\left(\frac{x}{a+2d}\right)^3\right\}\ldots$$

can be expressed in finite terms if x-2a be a multiple of d.

2. Suppose now that $\alpha=\beta$ in (5). We obtain

$$\left\{1+\left(\frac{2\alpha}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{2+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{3+\alpha}\right)^3\right\}$$

$$=\frac{\{\Gamma(1+\alpha)\}^3\sinh\pi\alpha\sqrt{3}}{\Gamma(1+3\alpha)\pi\alpha\sqrt{3}}. \quad \ldots(6)$$

Similarly, putting $\beta=\alpha+1$ in (5) we obtain

$$\left\{1+\left(\frac{2\alpha+1}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha+1}{2+\alpha}\right)^3\right\}\ldots$$

$$=\frac{\{\Gamma(1+\alpha)\}^3\cosh\pi(\tfrac{1}{2}+\alpha)\sqrt{3}}{\Gamma(2+3\alpha)\pi}. \quad (7)$$

Again, since

$$\left\{1+\left(\frac{\alpha}{n}\right)^3\right\}\left\{1+3\left(\frac{\alpha}{2n+\alpha}\right)^2\right\}=\frac{(1+\frac{\alpha}{n})(1+\frac{\alpha^2}{n^2}+\frac{\alpha^4}{n^4})}{(1+\frac{\alpha}{2n})^2},$$

it is easy to see that

4. It is also easy to see that

$$\frac{1^2}{1^3+n^3}-\frac{2^2}{2^3+n^3}+\frac{3^3}{3^3+n^3}-\frac{4^3}{4^3+n^3}+\cdots$$

$$=\frac{1}{3}\left(\frac{1}{1+n}-\frac{1}{2+n}+\frac{1}{3+n}-\frac{1}{4+n}+\cdots\right)$$

$$+\frac{4}{3}\left\{\frac{2-n}{(2-n)^2+3n^2}-\frac{4-n}{(4-n)^2+3n^2}+\frac{6-n}{(6-n)^2+3n^2}-\cdots\right\} \quad \ldots(13)$$

Since $\dfrac{\pi}{4\cosh \frac{1}{2}\pi x}=\dfrac{1}{1^2+x^2}-\dfrac{3}{3^2+x^2}+\dfrac{5}{5^2+x^2}-\cdots,$

it is clear that the left-hand side of (13) can be expressed in finite terms if n is any odd integer. For example,

$$\frac{1^2}{1^3+1}-\frac{2^2}{2^3+1}+\frac{3^2}{3^3+1}-\frac{4^2}{4^3+1}+\cdots$$

$$=\frac{1}{3}(1-\log 2+\pi \operatorname{sech} \tfrac{1}{2}\pi\sqrt{3}). \quad \ldots(14)$$

The corresponding integral in this case is

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\cdot\frac{dx}{n^6+x^6}=\frac{2}{\pi}\int_0^\infty\left\{\frac{1}{2x^2}+\sum_{\nu=1}^{\nu=\infty}\frac{(-1)^\nu}{\nu^2+x^2}\right\}\frac{x^6dx}{n^6+x^6}$$

$$=\frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}+\cdots\right)$$

$$-\frac{4}{3}\left\{\frac{n+2}{(n+2)^2+3n^2}-\frac{n+4}{(n+4)^2+3n^2}+\frac{n+6}{(n+6)^2+3n^2}-\cdots\right\}; \quad \ldots(15)$$

and so the integral on the left hand side of (15) can be expressed in finite terms if n is an odd integer. For example,

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\cdot\frac{dx}{1+x^6}=\frac{1}{3}(\log 2-1+\pi \operatorname{sech} \tfrac{1}{2}\pi\sqrt{3}). \quad \ldots(16)$$

$$\left[\left(1+\frac{a^3}{1^3}\right)\left(1+\frac{a^3}{2^3}\right)\ldots\right]\left[\left\{1+3\left(\frac{a}{2+a}\right)^2\right\}\left\{1+3\left(\frac{a}{4+a}\right)^2\right\}\ldots\right]$$

$$=\frac{\Gamma(\frac{1}{3}a)}{\Gamma\{\frac{1}{3}(1+a)\}}\left(\frac{\cosh\frac{\pi a\sqrt{3}-\cos\pi a}{2}}{\frac{a+2}{3}}\right). \qquad \ldots(8)$$

3. It is known that, if the real part of a is positive, then

$$\log\Gamma(a)=(a-\tfrac{1}{2})\log a-a+\tfrac{1}{2}\log 2\pi+2\int_0^\infty \frac{\tan^{-1}(x/a)}{e^{2\pi x}-1}\,dx. \qquad \ldots(9)$$

From this we can shew that, if the real part of a is positive, then

$$\tfrac{1}{2}\log 2\pi a+\frac{\pi a}{\sqrt{3}}+\log\left\{\left(1+\frac{a^3}{1^3}\right)\left(1+\frac{a^3}{2^3}\right)\left(1+\frac{a^3}{3^3}\right)\ldots\right\}$$

$$=\log\left(\frac{\cosh\frac{\pi a\sqrt{3}-\cos\pi a}{3}}{\pi a}\right)+2\int_0^\infty \frac{\tan^{-1}(x/a)^3}{e^{2\pi x}-1}dx. \qquad \ldots(10)$$

From this and the previous section it follows that

$$\int_0^\infty \frac{\tan^{-1}x^3}{e^{2\pi nx}-1}dx$$

can be expressed in finite terms if n is a positive integer. Thus for example,

$$\int_0^\infty \frac{\tan^{-1}x^3}{e^{2\pi x}-1}\,dx=\frac{1}{4}\log 2\pi-\frac{\pi}{4\sqrt{3}}-\frac{1}{2}\log(1+e^{-\pi\sqrt{3}}); \qquad \ldots(11)$$

$$\int_0^\infty \frac{\tan^{-1}x^3}{(e^{4\pi x}-1)}\,dx=\frac{1}{8}\log 12\pi-\frac{\pi}{4\sqrt{3}}-\frac{1}{4}\log(1-e^{-2\pi\sqrt{3}}); \qquad \ldots(12)$$

and so on.

1

Ramanujan's Contributions to the theory of Elliptic Functions

by
K. Venkatachaliengar

Ramanujan's Note Books 1,2 (and 3) were made available to scholars as lithographed copies in 1957. The originals of these can be traced back to 1910. Ramanujan had an attack of T.B. in 1910 at Madras; one of his sincere friends Sri Radhakrishnan who later became an advocate in Tanjavour (Tamil Nadu) requested Doctor Narayanswamy to examine him. On his advice Radhakrishnan sent Ramanujan to his home town Kubhakonam for needful rest and careful nursing. At that time Ramanujan gave two big sized note books to his friend for safe keeping and told him that in case Ramanujan died the Note books were to be handed over either to Prof. Singaravelu Mudaliar of Pachaiappa's College or Prof. Ross of Christian College who could appreciate the greatness of Ramanujan; fortunately Ramanujan recovered completely.

Till 1910 Ramanujan had seen only books like Carr's *Synopsis of results in pure mathematics* and perhaps one or two books on Calculus. [Williamson's and Edwards *Integral Calculus*] He became a research scholar of the Madras University in 1913,

about that time the University Library was established. It did contain advanced mathematical treatises—

(1) Tannery and Molk : *Functiones Elliptiques 1* to *4*

(2) Whittaker and Watson :*Modern Analysis*

(3) Greehill, : *Elliptic Functions*

(4) Cayley : *Elliptic Functions*

(5) Mathews : *Theory of Mumbers 1* to *2*

etc. Perhaps Ramanujan could have been allowed to use these. A detailed examination of his two note books reveals the fact that he did not look at any of these books. This will be made clearer in what follows.

More than two-thirds of the note book [The two are almost identical] concerns itself with the development of the formulae of Elliptic Function Theory and the transcendentals arising from the theory and relations among them. If Ramanujan had seen either Greenhill or Cayley he would have recognised that Elliptic functions are doubly-periodic. Nowhere do we find in the Note Book double periodicity mentioned. There are no double series or products. Even if a function occuring in his note book displays double periodicity he is not mindful of it at all. The addition theorem is nowhere mentioned, except as problems in Integral Calculus concerning elliptic integrals of the first kind in Legendre form. If he had seen Greenhill or Cayley's book the addition theorem of the Jacobian elliptic functions of sn (u) cn (u). dn (u) in the various beautiful forms given by Cayley would have certainly fascinated Ramanujan, and Ramanujan would have enriched them further certainly. These functions sn (u), cn (u), dn (u), [with Ramanujan's own notation for these] are defined by their Fourier series developments and only the inter-relations concerning these and their derivates are derived There is no attempt made to connect them with the elliptic integrals of the first kind given elsewhere in the note book. The Weierstrassian functions $P(z)$, $\xi(z)$, $\sigma(z)$ do not occur at all in the note book explicitly. The Eisenstein functions occuring as coefficients of expansion connected with these functions do occur in the note books; in fact these

together wirh their mutual relations, their imaginary transformations, quadratic transformation do occur profusely in the note book. We shall systematically give an account of these and connected functions later on.

Another evidence to show that Ramanujan had not seen the above mentioned books is the completely different unique symbology and notation adopted by him in the note book. The Legendre constants k and K are replaced by Ramanujan with $x = k^2$, and $Z = K$. The Weierstrassian invariants g_2 and g_3 have been denoted by M, N [or Q,R later on] and the constant η has been denoted as L [later on P].

Ramanujan has developed θ functions of Jacobi in detail with his own peculiarly unique notation. The famous Jacobi parameter q is not at all found in the note book. The Jacobi functious θ (u) and H (u) or the more modern $\theta_1, \theta_2, \theta_3, \theta_4$ do not occur in the note book at all. The term *modular equation* does not occur in the note book [even though Ramanujan has given numerous forms of modular equations of several degrees profusely in the note book] [In exercise 15 (i) p. 191-192 Note book 2. Ramanujan calls some relations as modular equations; these deal with an elementary topic and have nothing to do with the more involved ones of elliptic function theory]. For instance the Modular equation of degree 3 in Legendre form occurs in the note book in the following ing form :

If β is in the third degree of a, then

$$\sqrt[4]{a\beta} + \sqrt[4]{(1-a)(1-\beta)} = 1$$

$$\text{or } \sqrt[8]{\frac{a^3}{\beta}} - \sqrt[8]{\frac{(1-a)^3}{1-\beta}} = \sqrt[8]{\frac{(1-\beta)^3}{1-a}} - \sqrt[8]{\frac{\beta^3}{a}} = 1$$

$$\text{or } \sqrt{a(1-\beta)} + \sqrt{\beta(1-a)} = 2\sqrt[8]{a\beta(1-a)(1-\beta)}$$

[Note Book 2 p. 230]

After Ramanujan went to England in 1914 he has published two papers.

with the multiplication of the partial fraction development of circular functions such as cot (z) , csc (z), etc; the series involved are absolutely convergent. The method is explained fully in Ex(18) of this chapter. In excercise (19) occurs the expansion :
$\pi^2 xy \cot(\pi x) \coth(\pi y) =$

$$1+2\pi xy\left[\frac{\coth\frac{\pi x}{y}}{1^2+y^2}+\frac{\coth\frac{2\pi x}{y}}{2^2+y^2}+\frac{\coth\frac{3\pi x}{y}}{3^2+y^2}+\ldots\text{to }\infty\right]+$$

$$2\pi xy\left[\frac{\coth\frac{\pi}{x}}{1^2-x^2}+\frac{\coth\frac{2\pi y}{x}}{2^2-x^2}+\frac{\coth\frac{3\pi y}{x}}{3^2-x^2}+\ldots\text{to }\infty\right] \qquad (1)$$

+ a logarithmic term

This and the other numerous expansions of this chapter can also be obtained by residue calculus. It is a matter of surprise that none of the classical treatises of complex function theory contain these expansions of Ramanujan although these form a very good collection of problems to illustrate Canchy's method of obtaining partial fraction developments using residue-calculus. As a particular case of the above formula (1) Ramanujan gives the formula

If $\alpha\beta=\pi^2$

$$\frac{\pi}{2}\coth\sqrt{u\alpha}\coth\sqrt{u\beta}=\text{a logari hmic term}$$

$$+\frac{1}{2u}+\sum_1^\infty\left[\frac{m\alpha\coth(m\alpha)}{u+m^2\alpha}+\frac{m\beta\coth(m\beta)}{u-m^2\beta}\right] \qquad (2)$$

expanding both sides in powers of $u=$ and comparing coefficient Ramanujan obtains

If $\alpha\beta=\pi^2$

$$\alpha\left[\frac{1}{e^{2\alpha}-1}+\frac{2}{e^{4\alpha}-1}+\frac{3}{e^{6\alpha}-1}+\ldots\text{to }\infty\right]+$$

$$+\beta\left[\frac{1}{e^{2\beta}-1}+\frac{2}{e^{4\beta}-1}+\frac{3}{e^{6\beta}-1}+\ldots\text{to }\infty\right]$$

$$=\frac{\alpha+\beta}{24}-1 \qquad (3)$$

S. Ramanujan *Collected Papers* [Camb. Univ. Press]

(6) Modular Equations and approximations to π.
Quart. Jour. Math. 1914. p.23 *et seq.*

(18) On certain Arithmetical functions
Trans. Camb. Phil. Soc. p. 136 *et seq.*

In these Ramanujan adopts the usual notation and gives the connection between his and the standard notation.

It is only in the above paper (6) that the relevant term Modular Equation is employed by him in the usual sense.

We proceed to describe briefly the results developed by him in his note book in connection with the theory of Elliptic functions. First of all it must be noted that Ramanujan's developement is purely algebraic (Comparison of coefficients of two equivalent power series etc] No Cauchy theory is employed. Occasionally Fourier.-Double Integral Theorem is used. It may be pertinent to mention in this connection that Jacobi also developed the theory in a purely algebraical way and his student Borchard has worked out the details of the theory. This is not so simple as Ramanujan's.

In chapter 14, Note Book 2, Ramanujan gives a number of formulae of the theory of elliptic functions. The order in which the formulae occur is not a logical one naturally; Ramanujan seems to have copied them from some of his other note books in which they were obtained roughly at first. It is strange to observe that the imaginary transformation of the Dedekind η-function and the Eisenstein (Weierstassian) functions occur at the outset. No elliptic functions occur in this chapter. Ramanujan was an expert in Fourier-integral and Mellin-integral transformations, he seems to have obtained the imaginary transformation formulae of the elliptic transcendentals in this way also. The imaginary transformations of certain allied functions occuring in his paper [Some definite integrals-Jour. *Ind. Math. Soc. 1919-p. 81–87,* Collected Papers p. 202 et et seq.] show the very effective way in which he uses these classical transformations. But in this Chap XIV of the note book he uses far more elementary means. His method starts

This in another form given below is the imaginary transformation of the Dedekind η-function; in Ramanujan notation:

$$\text{If } q=\exp(2\pi i\tau),\ p=\exp\left(-\frac{2\pi i}{\tau}\right)$$

$$P(q)=1-24\sum_{1}^{\infty}\frac{nq^n}{1-q^n}\ ;\ \text{then} \tag{4}$$

$$P(p)=\tau^2 P(q)+\frac{6\tau}{\pi i}$$

Integrating this w.r.t. τ and taking exponentials we obtain the im ginary transformation of the Dedekind functions. This is the most elementary method. One has to thank Ramanujan for this simple method, which can be in corporated in our text books. Ramanujan has also remarked that these expansions can also be obtained by Residue calculus. Unaware of Ramanujan's work Siegel has published in *Acta Arithemtica* [see Siegel *Collected works* Bd. 3.p.188] his one page proof of this theorem by Residue calculus; this proof is a model for crispness and clarity. Serre in his excellent book *Caurse of Arithemetic* has given a simple proof using the double series expansion of Eisenstein-Weierstrss. It is rather strange that some recent books [Apostol-*Analysis*-K.G. Ramanathan *J I M.S.* Vol.46 p.111] consider the imaginary transformation of a more general function due to Hecke which can be ob ained by complicated contour integration and taking limits with respect to a parameter.

In this connection Ramanujan obtains comparing coefficients in the expansion in powers of u on both sides of (2) the imaginary transformations of the more involved functions in Note Book 2. Ch. 14, Ex. 21.

If $\alpha\beta=\pi^2$, and $n>1$ $n=1$ also by a slight modification]

$$(4\alpha)^{1-n}\left[\tfrac{1}{2}S_{2n-1}+\sum_{r=1}^{\infty}\frac{1}{r^{2n-1}\left[e^{2r\alpha}-1\right]}\right]+$$

$$(-4\beta)^{1-n}\left[\tfrac{1}{2}S_{2n-1}+\sum_{r=1}^{\infty}\frac{1}{r^{2n-1}\left[e^{2\pi\beta}-1\right]}\right]$$

$$=\frac{B_{2n}}{(2n)}\left[(-\alpha)^n+\beta^n\right]+\pi^2\frac{B_2\ B_{n-2}}{2!\,(2n-2)!}\cdot\left[(-\alpha)^{n-2}+\beta^{n-2}\right]$$

$$+\ldots[\text{a finite number of term}]\,, \qquad (5)$$

$$S_{2n-1}=\sum_{1}^{\infty}\frac{1}{r^{2n-1}}$$

B_{2r}–Bernoulli number, notation as in Note book 2 of Ramanujan.

The functions involved in the left hand side of (2) are not the Eisenstein Weierstrass functions which involve terms like

$$\Sigma_{2r-1}\sigma(n)\,q^n\,,r\geqslant 1$$

but involve terms like

$$\Sigma_{2r-1}\sigma(n)\,q^n\,,\ r\geqslant -1.$$

These imaginary transformations are new due to Ramanujan, they are obtained in such a simple way. They can also be obtained by Residue Calculus apparently; and by repeated differetiation of the Ramanujan relations (5). One can obtain the imaginary transformations of the usual Eisenstein-Weierstrass functions. The other formulae given by Ramanujan in this chapter relate to those obtained by him by the use of his famous identity. Some deductions from this identity are in this chapter; the expressions for the higher order Eisenstein functions as polynomials in the Ramanujan function : Q, R :

$$Q=1+240\sum_{1}^{\infty}\frac{n^3q^n}{1-q^n}\,,\ R=1-504\sum_{1}^{\infty}\frac{n^5q^n}{1-q^n} \qquad (6)$$

are also given in this chapter along with their imaginary transformations, with a large number of particular cases, taking $q=e^{-\pi}$ or $e^{-2\pi}$.

The Ramanujan Differential equations :

$$q\frac{dP}{dq}=\frac{P^2-Q}{12},\quad q\frac{dQ}{dq}=\frac{PQ-R}{3},\quad q\frac{dR}{dq}=\frac{PR-Q^2}{2} \tag{7}$$

along with the very simple deduction of the Dicriminant product identity of Jacobi is also in this chapter :

$$Q^3-R^2=1728q\prod_1^\infty(1-q^n)^{24} \tag{8}$$

we give here Ex 13 (X) [Note book 2 p. 190] to illustrate Ramanujan's Lore of Large numbers.

$$7709381041219+32640\sum_1^\infty\frac{n^{11}\,q^n}{1-q^n}=$$

$$=764412173217Q^8+5323905468000Q^5R^2+ \\ 16210034000000Q^2R^4 \tag{9}$$

Q and R in terms of Weierstrassian invariants :

$$Q=12g_2,\ R=215g_3 \tag{10}$$

The proofs are not indicated in this chapter. They are obtained by the use of one of the identities of Ramanujan given explicitly in one of his published papers. We shall deal with this in detail in the next paragraph.

Ramanujan published two papers on topics in Elliptic founction theory :

Modular equations and approximations to π
[*Collected Papers* p. 23 to 39]

(2) on certain Arithmetical functions
[*Collected Papers* p.136–162]

In these papers he explains his theorems with the standard notations of the theory of elliptic functions. [one of the periods of Ramanujan is always 2π or π. He never notices the double

periodicity]. The first of these papers contains a proof of the determination of the singular moduli corresponding to 5 by means of the corresponding modular equation...It contains without proof the expressions in terms of radicals of a large number of singular moduli [The editor asked him to omit those given by Weber], a number of interesting series regarding π [which are also found in the note book partially] Some of them without proof and some with proofs. One interesting result he proves [with a large number of particular cases] is the following :

$$\text{If } 1-24\sum_{1}^{\infty}\frac{nq^n}{1-q^n}=P(q), \text{ then } mP(q^m)-P(q)$$

[m, a natural number] is a rational polynomial in α,β, where α is the Legendre invariant k^2 and β is the one obtained from α by replacing q by q^m. This is due to Hermite and occurs only in advanced treatises on the topic. The next paper *On certain Arithmetical functions* contains his famous identity :

$$\left\{\frac{1}{4}\cot\frac{\theta}{2}+\sum_{1}^{\infty}\frac{x^n\sin(n\theta)}{1-x^n}\right\}^2=\left[\frac{1}{4}\cot\frac{\theta}{2}\right]^2+ \qquad (11)$$

$$+\sum_{1}^{\infty}\left[\frac{x^n\cos(n\theta)}{(1-x^n)^2}+\frac{1}{2}\frac{nx^n(1-\cos n\theta)}{(1-x^n)}\right]$$

He gives a complete proof of this identity and indicates how another identity can be proved in a similar manner. See [*Collected Papers* p.139 formula (18)].

Ramanujan makes these identities as foundation for his development of Elliptic function theory. The second of these is equivalent to the differential equation of Weierstrassian function $\wp(u)$

$$\wp''(u)=\wp^2(u)-\frac{g_2}{2} \qquad \wp''(u)=6\wp^2(u)\,g^2/_2 \qquad (12)$$

He obtains some very interesting new results. The basic one is his differential equation mentioned earlier

$$q\frac{dP}{dq}=\frac{P^2-Q}{12}\ ,\ q\frac{dQ}{dq}=\frac{PQ-R}{3}\ ,\ q\frac{dR}{dq}=\frac{PR-Q^2}{2}$$

$$P = 1-24 \sum_{1}^{\infty} \frac{nq^n}{1-q^n}, = 1+240 \sum_{1}^{\infty} \frac{n^3q^n}{1-q^n} , \qquad (13)$$

$$R = 1-504 \sum_{1}^{\infty} \frac{n^5q^n}{1-q^n}$$

These were given by him *[Collected Papers* p.142 formula (30)] with complete proof. [Note N.M. Katz, *Annals of Mathematics* 104 (1976) gives these differential equations with a more complicated proof using Hurwitz's method of Modular functions *with no reference* to the very simple method of Ramanujan. He however calls the functions P,Q,R, as Ramanujan functions.]

Ramanujan deduces the famous Discriminant product identity of Jacobi by logarithmically differentiating the product and using trivially his differential equations [*Collected Papers*–p.144 formulae 42-44].

Ramanujan deduces from his identities:

(1) The higher order Eisenstein functions are polynomials in Q,R over the rational field. [which is a result of Weierstranss]

(2) $\sum_{(m,n)=1}^{\infty} m^r n^s q^{mn}$ [r and s positive integers (r+s) being odd] (14)

is a polynomial in his functions, P,Q,R over the rational field. He has also given a large number of particular cases and deduced some results of higher arithmetic. The result in (14) has been rediscovered by Andre Weil in his recent book *Elliptic functions according to Eisenstein and Kronecker*-(Springer) Ch V variation I p.45 §1 and 2.

(There is no reference to Ramanujan in this connection in the book].

The basic identity of Ramanujan (13) is found in earlier literature. It is contained in Jordan–*Cours da Anal.* T.2, but it is contained after about 150 pages of elliptic function theory development. And one notices that it can be proved by noting that the Ramanujan function [Left side of (11)] is the logarithmic derivate

of a θ function of Jacobi and Ramanujan's identity is an easy deduction when we use the parabolic partial differentiation of the θ-function. The remarkable thing is Ramanujan takes this identity as basic and shows how the elliptic functions can be developed -a priori-. The proofs are simpler e.g. the proof of the Discriminant Product Identity.

The same paper [*Collected Papers* p.147 formula 665), contains four identities the first two of which are the classic ones of Euler and Jacobi but the other two are Ramanujan's own. These two are

$$\frac{f^5(q)}{f^2(q^2)} = q^{\frac{1^2}{24}} - 5q^{\frac{5^2}{24}} + 7q^{\frac{7^2}{24}} - 11q^{\frac{11^2}{24}} + - \qquad (15)$$

$$\frac{f^5(q)}{f^2(-q)} = q^{\frac{1^2}{3}} - 2q^{\frac{2^2}{3}} + 4q^{\frac{4^2}{3}} - 5q^{\frac{5^2}{3}} + -$$

(1,2,4,5,... are natural numbers excluding multiples of 3, and 1,5,7,11,.... are odd numbers excluding multiples of 3 and

$$f(q) = q^{\frac{1}{24}} (1-q)(1-q^2)(1-q^3)\ldots \text{ to } \infty$$

The basic identity of Ramanujan is easily capable of a generalisation, and the proof in the general case is very much simpler than in the particular case. The general identity is : [$|q|<1$].

Let

$$P_1(z) = \tfrac{1}{2} + \sum_{-\infty}^{\infty}{}' \frac{z^n}{1-q^n}, \quad P_2(z) = -\frac{1}{12} + \sum_{-\infty}^{\infty}{}' \frac{q^n z^n}{(1-q^n)^2} \qquad (16)$$

(where the dash indicates that $n=o$ is excluded in the summation], then

$$P_1(\alpha)P_1(\beta) - P_1(\alpha\beta)\,[P_1(\alpha)+P_1(\alpha)] = P_2(\alpha) + P_2(\beta) + P_2(\alpha\beta). \qquad (17)$$

The Laurent expansions in (16) are naturally valid for $|q|<|z|<1$. We first of all assume that $|q^{\frac{1}{2}}|<\alpha,\beta<1$, which implies $|q|<|\alpha\beta|<1$, also let the Laurent expansion of (17) be

$$\sum_{(m,n)} C_{m,n}\, \alpha^m \beta^n$$

Note that the left side of (17) consists of three products of two terms each.

Case (i) Let $mn(m-n) \neq 0$, then

$$C_{m,n} = \frac{1}{(1-q^m)(1-q^n)} + \frac{1}{(1-q^m)(q^{n-m}-1)} - \frac{1}{(1-q^n)(1-q^{m-n})}$$

$$= \frac{(1-q^{m-n})\times(1-q^n)\,q^{m-n} - (1-q^m)}{(1-q^m)(1-q^n)(1-q^{m-n})} = 0 \text{ evidently.}$$

Case (ii) $m=n=0$

$$C_{o,o} = \frac{1}{4} - \frac{1}{2.2} - \frac{1}{2.2} = -\frac{1}{4}$$

Case (iii) $m \neq o$, $n=0$

$$C_{m,o} = -\frac{1}{2(1-q^m)} - \frac{1}{2(1-q^m)} + \frac{1}{(1-q^m)(1-q^m)} = \frac{q^m}{(1-q^m)^2}$$

similarly $C_{o,n} = \dfrac{q^n}{(1-q^n)^2}$

$n \neq 0$

Case (iv) $m=n \neq 0$

$$C_{m,m} = \frac{1}{(1-q^m)(1-q^m)} - \frac{1}{2(1-q^m)} - \frac{1}{2(1-q^m)} = \frac{q^m}{(1-q^m)^2}$$

these exhaust all cases.

[The significant thing is the bulk of the terms in the left side viz those corresponding to

$mn(m-n) \neq 0$, are all xero.

Substituting we obtain the identity (17)]

Expanding each term on right side of $\rho_1(z)$ and $\rho_2(z)$ in powers of q and rearranging the terms we obtain

$$\rho_1(z) = \frac{1}{2}\left(\frac{1+z}{1-z}\right) + \sum_1^\infty \left[\frac{q^n z}{1-q^n z} - \frac{q^n z^{-1}}{1-q^n z^{-1}}\right] \tag{18}$$

$$\rho_2(z) = -\frac{1}{12} + \sum_1^\infty n\left[\frac{q^n z}{1-q^n z} + \frac{q^n z^{-1}}{1-q^n z^{-1}}\right]$$

We also easily obtain

we write $z=\exp(i\theta)$, $\alpha=\exp(ia)$, $\beta=\exp(ib)$ etc and define the functions

$$2i\phi_1(\theta)=\rho_1(z),\ 2\phi_2(\theta)=\rho_2(z)$$

The identity satisfied by $\phi_1(\theta)$, $\phi_2(\theta)$ is

$$-\frac{1}{4}\phi(a)\phi_1(b)+\frac{1}{4}\phi(a+b)[\phi_1(a)+\phi_1(b)]=$$

$$=-\frac{1}{4}+2[\phi_2(a)+\phi_2(b)+\phi_2(a+b)]$$

or in another from, if $a+b+c=0$

$$\phi_1(a)\phi_1(b)+\phi_1(b)\phi_1(c)+\phi_1(c)\phi_1(a)= \quad (2)$$
$$=1-8[\phi_2(a)+\phi_2(b)+\phi_2(c)]$$

Taking limits as $b\to a$ we obtain

$$\phi^2{}_1(a)+\tfrac{1}{2}\phi_1(a)=\phi_2+\tfrac{1}{2}\phi_2(0) \quad (21)$$

This is the same as Ramanujan's identity (11) The Weierstrassian functionp (a) is

$$p(a)=2[\phi_2(0)-\phi_1'(a)] \quad (21)$$

It is easily verified that p(a) is doubly periodic, the periodlattice spanned by

$$(2\pi,\ 2\pi\tau)$$

where $x=\exp(2\pi i\tau)$

x is the same as the parameter q of Jacobi. From relations (20) one easily derives the addition theorem of p(a) in the Abelian as well as in the other form.

$$\begin{vmatrix} 1 & 1 & 1 \\ p(a) & p(b) & p(c) \\ p(a) & p'(b) & p'(c) \end{vmatrix} =0,\ a+b+c=0 \quad (22)$$

$$p(a+b)=\frac{3}{4}\left[\frac{p'(a)-p'(b)}{p(a)-p(b)}\right]^2-p(a)-p(b)$$

Ramanujan never gives the Addition Theorem since he had not seen any book on Elliptic function. Ramanujan derives the differential equation by employing another identity as already mentioned which can also be derived from the generalised identity (20). At the end of his paper. On certain Arithmetical functions, *Collected Works* p.151 et seq. he defines his function $\tau(n)$:

$$\sum_{1}^{\infty} \tau(n)\, q^n = q[(1-q)(1-q^2),\ldots,(1-q^n)\ldots]^{24}$$

and the corresponding Dirichlet function satisfies the product formula

$$\sum \frac{\tau(n)}{n^s} = \prod_{p \text{ (all primes)}} (1-\tau p).\, p^s + p^{11-2s})^{-1}$$

and made his famous conjecture

$$|\tau(n)| < n^{\frac{11}{2}+\epsilon} \text{ for every } \epsilon > 0$$

which was proved by Deligne sixty years later.

Although Ramanujan starts elliptic function theory with a function related to the Weierstrassian function p(u) he does not mention anywhere the roots e_1, e_2, e_3, of the related cubic. The basic functions which are important to him are the functions P,Q,R mentioned earlier. We recall the expansion

$$P(q) = P = 1 - 24 \sum_{1}^{8} \frac{n\, q^n}{1-q^n}$$, he writes

$[4P(q^2)-2P(q)]$; $[P(q^{\frac{1}{2}})-2P(q)]$ for the classical quantities $12e_1$ and $12e_3$ respectively. Later he obtains all the concerned product expression and the relevant Lambert series for these e.g.

$$e_2-e_3 = 4h\prod' [(1-h^{2n})^4\ (1+h^{2n})^8]$$
$$4(e_1-e_2) = \prod' [(1-h^{2n})^4\ (1-h^{2n-1})^8] \qquad (23)$$
$$h = \exp(\pi i t) = q^{\frac{1}{2}}$$

Ramanujan has exploited the quadratic transformation to the greatest possible extent, we recall that Hermite in one of his papers has expressed such a wish.

We have mentioned earlier that Ramanujan has obtained the imaginary transformation of P,Q,R, and the higher order Eisenstein functions by a simple elementary method to which attention has not been bestowed at all. If one wishes to treat these by

See Serre : *A course of Arithmetic* p.105 (bottom)

Using the generalised form of the Ramanujan identity one can obtain an explicit solution of the Halphen Differential equations;

u_1, u_2, u_3 are functions of $h = \exp(\pi i \tau)$ which satisfy the simultaneous differential equations $[D = = h\frac{d}{dh}]$

$$D(u_2+u_3) = u_2u_3, D(u_3+u_1) = u_3u_1, D(u_1+u_2) = u_1u_2$$

the solution first given by Halphen :

$$u_1 = 1 + 8\left[\frac{h^2}{1-h^2} - \frac{2h}{1-h^4} + \frac{3h^6}{1-h^6} \cdot + \ldots \text{ to } \infty\right]$$

$$u_2 = 8\left[\frac{h}{1-h^2} - \frac{2h^2}{1-h^4} + \frac{3h^3}{1-h^6} - . + \ldots \text{ to } \infty\right] \qquad (26)$$

$$u_3 = -8\left[\frac{h}{1-h^2} + \frac{2h^2}{1-h^4} + \frac{3h^3}{1-h^6} + \ldots \text{to } \infty\right]$$

these are useful in obtaining the classical formulae of the Elliptic function theory.

The simplest method of proof of the above is by means of the generalised Ramanujan identity. On page P.201 of his note book 2. Ramanujan has given these solutions in more or less the same form.

Ramanujan treats fully in an independent way the theory of the Jacobian θ-functions. His definition of the θ-function [ch 16 of his note book] in the following:

If $[\,|ab| < 1]$

$$f(a, b) = 1 + (a+b) + ab\,(a^2+b^2) + \ldots (ab)^{\frac{n(n-1)}{2}}(a^n+b^n) + \ldots \text{to} \infty$$

he notices the evident relations

$f(a,b) = f(b,a)$; $f(1,a) = 2f(a,a^3)$

$f(-1,a) = o$ and for any positive integer n

assuming complex function theory i.e. Lioville's classical theorem Ramanujan's function :

$\phi_1(\theta)$ or $\phi_1(\theta,\tau)$ in the following way one deduces easily that

$$\psi_1(\theta) = \phi_1(\theta,\tau) - \frac{1}{\tau}\phi_1\left(\frac{\theta}{\tau},-\frac{1}{\tau}\right) - \frac{\theta}{4\pi i}$$

is an analytic function with no singularities and which is doubly periodic $[2\pi, 2\pi\tau]$ which is therefore a constant which is easily verified to be equal to zero. Comparing coefficients in the Laurent expansion in u we can also obtain the quadratic transformation formulae :

If $q = \exp(2\pi i\tau)$, $q = \exp\left(-\frac{2\pi i}{\tau}\right)$, $\mathrm{Im}(\tau) > 0$ then

$$P(p) = \tau^2 P(q) + \frac{6\tau}{\pi i}$$

$$Q(p) = \tau^4 Q(q) \qquad (24)$$

$$R(p) = \tau^6 R(q)$$

Since the Ramanujan function $\phi_1(\theta)$ is the logarithmic derivate of a θ function we obtain the classical formula :

$$(1)\ \mathrm{Exp}\left(-\frac{a^2}{4\pi i\tau}\right)\frac{\sin\frac{a}{2}}{\sin\frac{a}{2\pi}}\prod_1^\infty\left[\frac{(1-q^n\alpha)(1-q^n\alpha^{-1})}{(1-p^n e^{-ia/\tau})(1-p^n e^{-ia/\tau})}\right] =$$

$$= i\,\mathrm{Exp}\left[\frac{\pi i}{6}(\tau - \tau^{-1})\right]$$

$$[\alpha = \exp(ia)] \qquad (25)$$

$$(2)\ \mathrm{Exp}\left(\frac{\pi i\tau}{12}\right)\prod_1^\infty(1-q^n) = \frac{1}{\sqrt{-i\tau}}\,\mathrm{Exp}\left(-\frac{\pi i}{\tau}\right)\prod_1^\infty(1-p^n)$$

(1) is the imaginary transformation of a θ function and (2) is that of the Dedekind η-function.

The function $\underline{Q}$ and $\underline{R}$ and the Jacabi-product $q\prod_1^\infty(1-q^n)^{24}$ can be used to form Integral basis for modular forms with integral coefficients;

$$f(a, b)=a^{\frac{n(n+1)}{2}}\, b^{\frac{n(n+1)}{2}}\, f[a(ab)^n, b(ab)^{-n}]$$

and the significant Jacobi product formula :

$$f(a,b)=\prod(a,ab)\prod(b,ab)\prod(-ab,ab)$$

where Ramanujan defines :

$$\prod(a,x)=\prod_{o}^{\infty}(1+ax^n).$$

For the proof of this he suggests a heuristic method (which perhaps cannot be justified at all) at first but immediately suggests the use of one of his identities occuring previously in the same chapter *16* of his note book p. 196.

Let $|x|<1$ and we define (temporarily)

$$P(t)=\prod_{n=o}^{\infty}(1+x^n t)$$

and let $\phi(z)$ be the Ramanujan function :

$$\phi(z)=\phi(z,\alpha,\beta)=\frac{P(z)\ P(z^{-1}x)}{P(\alpha z)\ P(\beta z^{-1})}$$

where α,β, are constants with

$|\alpha\beta\ x|<1$, then in the annular region $|\beta x|<|z|<|\alpha|^{-1}$

$\phi(z)=\phi(z,\alpha,\beta)=$

$$\phi_o(\alpha,\beta)\ \left\{1+\frac{1-\alpha}{1-\beta\ x}\ z+\frac{(1-\alpha)(x-\alpha)}{(1-\beta x)(1-\beta x^2)}z^2\right\}+\ldots.$$

$$\left\{\frac{(1-\alpha)(x-\alpha)(x^2-\alpha)}{(1-\beta x)(1-\beta x^2)(1-\beta x^3)}z^3+\ldots.\text{to}\infty\ \right\}+$$

$$+\left[\left\{\frac{1-\beta\ x}{1-\alpha x\ \ 2}+\frac{(1-\beta)(x-\beta)}{(1-\alpha x)(1-\alpha x^2)}\left(\frac{x}{2}\right)^2+\ldots.\text{to}\ \infty\ \right\}\right]$$

where

$$\phi_o(\alpha,\beta)=\frac{P(-\alpha x)\ P(-\beta x)}{P(-x)\ P(-\alpha\beta x)} \qquad (27)$$

This formula has been called by Hardy as a "Remarkable Identity with many parameters" and it is recently called indent-

ity. If $\beta=\alpha^{-1}$ it is a classical identity of Jacobi whose proof is quite simple. But the case when $\alpha=\beta=0$ which is easily seen to be admissible in the Ramanujan Identity (27) cannot be proved by Jacobi methods easily. One has to have recourse to some artifice to obtain the final step in the proof. But the proof of the Ramanujan Identity which is a generalisation of the classical Jacobi formula :

$$\sum_{-\infty}^{\infty} x^{n^2} z^n = \prod_{1}^{\infty} [(1-x^{2n})(1+x^n z)(1+x^n z^{-1})]$$

is simple and straight forward on account of the introduction of poles in the generalised Ramanujan function and the residue at any one of the poles is evidently a product.

It is a matter of great surprise that Jacobi celebrated for his rich collection of interesting identities did not discover this interesting formula; it was left to Ramanujan to discover.

In chapter 16 Ramanujan gives all the identities satisfied by θ-functions more or less. We have the classic identities of Euler and Jacobi.

$$\prod_{1}^{\infty} (1-x^n) = \sum_{-\infty}^{\infty} (-1)^n x^{3n^2-n} \quad \text{(Euler)}$$

$$\prod_{1}^{\infty} (1-x^n)^3 = \sum_{1}^{\infty} (-1^n)(2n+1) q^{\frac{n(n-1)}{2}} \quad \text{(Jacobi)}$$

and two other identities of the same type mentioned earlier no (15) [*Collected Works* of Srinivasa Ramanujan—p. 147 formula (65)].

Ramanujan uses his remarkable identity (27) to obtain the product formulae etc of the Jacobian elliptic functions; the square relations among these functions can be derived by the use of his basic identity (11).

The fundamental constants used by Ramanujan to develop his theory of elliptic function are the functions P,Q,R and he

uses his differential equations mainly in the continuation of the work. Q and R are effectively the Weierstrassian constant g_2, g_3 and P is related to the constant η_1 of the ξ-function of weierstran :

$$\xi(u+2\omega_1)-\xi(u)=2\eta_1 \text{ etc}$$

The imaginary transformation connecting $P(e^{2\pi i\tau})$ and $P(e^{-2\pi i/\tau})$ is equivalent to the relationof Weierstran $\eta_1\omega_2-\eta_2\omega_1=\frac{\pi i}{2}$.

[The corresponding relation in Jacobian functions is originally due to Legendre himself]. The constants concerning the Jacobian elliptic functions sn (u) etc are the ones originally due to Legendre k and K. where

$$K=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

It is interesting to find that Ramanujan has obtained these from the Weierstrassian constants Q and R [g_2 and g_3] in his own independent way as explained below. [see chapter *16* of the Note book].

He defines the constant *X* and *Z* which are the Legendre constants *k* and *K* of Jacobian elliptic functions as those connected algabraically with Q, R :

$$Q=1+240\sum_1^{\infty}\frac{n^3q^n}{1-q^n}=Z^4(1-x+x^2)$$

$$R=1-504\sum_1^{\infty}\frac{n^5q^n}{1-q^n}=Z^6\left((1+x)(1-2x)(1-\frac{x}{2})\right)$$

one verifies that

$$Q^3-R^2=\frac{27x^2(1-x)^2}{4}Z^{12}$$

And from Ramanujan differential equations

$$q\frac{d}{dq}\left[\text{Log}(Q^3-R^2)\right]=P$$

one obtains the further relations :

$$q\frac{dx}{dq} = x' = \frac{x(1-x)z^2}{2}$$

$$12\frac{z'}{z} = P - (1-2x)z^2$$

where the dash indicates differentiation w.r.t. log (q) utilising these one is lead to the Legendre equation :

$$4\frac{d}{dx}[x(1-x)z] - z = 0 \qquad (23)$$

We get

$Z = {}_2F_1(\frac{1}{2},\frac{1}{2};1;x)$ (Hypergeometric function) and proceeding further he obtains the classical definition of the Jacobi parameter q.

$$q = \exp\left(-2\pi\frac{K'}{K}\right) = \exp\left(-2\pi\frac{Z'}{Z}\right)$$

where

$$Z' = {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-x\right)$$

hence $Z = \frac{\pi K}{2}$, $Z' = \frac{\pi K'}{2}$

From the involution $x \leftrightarrow (1-x)$ one obtains afresh the imaginary transformation formulae.

As we have already mentioned the Halphen differential equations are found in Ramanujan's work [*Note book* 2 p. 201, ex 12 etc.] This enables him to find all the classical products and Lambert series of the theory in an interesting way.

The roots of the Weierstrassian cubic e_1, e_2, e_3 are not at all mentioned by Ramanujan any where in his work. He uses instead :

$$4(e_1 - e_2) = Z^2 \prod_1^{\infty} [(1-h^{2n})^4 (1+h^{2n-1})^8]$$

$$q^{\frac{1}{2}} = h = \exp(\pi i \tau)$$

$$P(-h)-P(q)=6e_2$$
$$2P(q^2)-P(q)=6e_1 \text{ etc}$$
$$P(h)-2P(q)=12e_3$$

Ramanujan has found that the function $Q(q^2)$, $R(q^2)$, $Q(h)$, $R(h)$ are also expressed in terms of X and Z (as simple polynomial expressions) similar to those of $Q(q)$ and $R(q)$; these are

$$Q(h)=Z^4(1+14x+x^2),\ R(h)=z^6(1-x)\ (1-34x+x^2)$$

$$Q\ (q^2)=Z^4\ (1-x+\frac{x^2}{16}),\ R\ (q^2)=Z^6\ (1-\frac{x}{2})\ (1-x-\frac{x^2}{32})$$

These have been worked out fully in his note book. This has not been noticed by others before him. These quadratic transformations combined with first order modular transformations of τ have enabled him to obtain all the interesting identities of the theory elliptic functions. He sets out in detail the effect of these transformations in the numerous examples of his note book. The method of obtaining these is by the the employment of his own fundamental differential equations.

Apart from these Ramanujan has obtained the connection of Q and R with three other hypergeometric equations. These have been mentioned in his paper-Modular equations and approximations to π [*Collected Works* p.37] and have been used by him to obtain several interesting series for $\frac{1}{\pi}$ (without proof in detail) [Ibid p. 37, 38]. These are : (1) Connection with the hypergeometric function ${}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)$. He defines in this connection; if

$$Q\ (y)=Q_1=1+240\ \sum_1^{\infty}\frac{n^3y^n}{1-y^n},$$

$$R\ (y)=R_1=1-504\ \sum_1^{\infty}\frac{n^5yn}{1-yn},$$

then x_1 and z_1 exist satisfying the follo ving relations

$Q_1=Z_1^4\ (1+3x_1)$; $R_1=Z_1^6\ (1-9x_1)$, then

$Q\ (y^2)=Z_1^4\ (1-\frac{3x_1}{4})$, $Z_1^6\ (1-\frac{9x_1}{8})=R\ (y^2)$ and obtains

$$Z_1 = {}_2F_1\left(\frac{1}{4}, \frac{3}{4} \; ; 1 \; ; x_1\right)$$

$$Z_1^2 = 1 + 24\left[\frac{y}{1-y} + \frac{3y^3}{1-y^2} + \frac{5y^5}{1-y^5} + \ldots\right]$$

and

$$y = \exp\left[-\frac{2\pi}{\sqrt{2}} \cdot \frac{{}_2F_1\left(\frac{1}{4}, \frac{3}{4} \; ; 1 \; ; 1-x_1\right)}{{}_2F_1\left(\frac{1}{4}, \frac{3}{4} \; ; 1 \; ; x_1\right)}\right]$$

One can show that $z_1^2 = 6e_1$ in Weierstrassian notation. Their relations are deduced by using Ramanujan Differential equations.

Ramanujan has remarked in the margin of his note book that X used by him earlier is related to x_1 by the simple relation :

$$x = \frac{2\sqrt{x_1}}{1+\sqrt{x_1}}$$

In this connection Ramanujan in one of his letters to Hardy from India *[Collected Works* p.353 Ex (19)] sent the following example :

If $\dfrac{{}_2F_1\left(\frac{1}{4}, \frac{3}{4} \; ; 1 \; ; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{4}, \frac{3}{4} \; ; 1 \; ; \alpha\right)} = \phi(\alpha) = 5\phi(\beta)$ then the following algebraical relation holds between α and β .

$$(\alpha\beta)^{\frac{1}{2}} + (1-\alpha)(1-\beta)^{\frac{1}{2}} + 8\left[\alpha\beta(1-\alpha)(1-\beta)\right]^{\frac{1}{6}} \left[(\alpha\beta)^{\frac{1}{6}} + \left[(1-\alpha)(1-\beta)\right]^{\frac{1}{6}}\right]$$

Littlewood after examining this exclaims in his letter to Hardy. "He (Ramanujan) is at least a Jacobi"

This is found in the collection of letters of

Hardy preserved in the Mathematical Library of Trinity College, Cambridge.

The other two hypergeometric functions which are connected with elliptic function theory by Ramanujan are

$${}_2F_1\left(\frac{1}{6},\frac{5}{6}\ ;\ 1\ ;\ \alpha\right) \text{ and } {}_2F_1\left(\frac{1}{3},\frac{2}{3}\ ;\ 1\ ;\ \alpha\right).$$

Several very interesting new identities are found by Ramanujan specially with the latter. In this connection a new elliptic function is obtained by Ramanujan.

Jacobian Elliptic functions sn (u) cn (u) and dn (u) have been treated fully by Ramanujan. All the relations between them are obtained; but there is no mention of the Addition theorem except in a very simple particular case. The general case is treated elsewhere in the note book only as an exercise in advanced integral calculus.

Ramanujan makes the interesting observation that the Legendre modular function $\lambda(z)$ can be defined uniquely by its quadratic transformation identity :

$$\lambda\ (o)=o,$$

$$\lambda\ (z)=\frac{4\sqrt{\lambda\ (m^2)}}{[1+\sqrt{\lambda\ (z^2)})]^2}$$

This enables one to deduce all the properties of $\lambda\ (z)$ and its inverse function fully and the classical Picard's Theorem can be proved in a simple way.

Ramanujan has obtained a huge number of Modular equations in several irrational forms analogous to those given by Schlafli. He has not given out his method anywhere except in the case of 3,5,7. For *3* he seems to have derived the equation by establishing identities concerning the products involved, and for *5* and 7 he has employed the classical identities of Euler and Jacobi which he employs in establishing his congruence relations concerning the partition function P(n). [Pages 235 et seq. of his note book2]. But for higher values of *n* the method used by him is not found

for $q = -e^{-\pi\sqrt{11}}, -e^{\pi/\sqrt{19}}, \ldots, -e^{-\pi\sqrt{163}}$

the values given by him are these :

11—$539Q^3 - 512R^2 = 0$

19—$(8^3+1)Q^3 - 8^3R^2 = 0$

27—$(40^3+9)Q^3 - 40^3R^2 = 0$ 27 not a prime.

43—$(80^3+1)Q^3 - 440^3R^2 = 0$

67—$(440^3+1)Q^3 - 440^3R^2 = 0$

163—$(533603+1)Q^3 - 533603R^2 = 0$

For 163 one has calculated recently $-\frac{1728Q^3}{Q^3 - R^2}$

$= 262, 537, 412, 64068000.$

The left side integer in Ramanujan's calculation is expressed as 533605.

Ramanujan's intuition tells him that there are no integers >163, with class number$=1$. This has been proved to be true a few years back.

K. Venkatachaliengar
8, II crows, Nehru Nagar
Bangalore 560020

at all inspite of efforts by G.N. Watson and others. There are very many modular relations other then classical one connecting the Legendre Modular function $\lambda(z)$ and $\lambda(z^n)$. Ramanujan has noticed that the quadratic transformation defines the Legendre-modular function $\lambda(z) = 16z + a_2z^2 + \ldots$, uniquley hence it must be possible to establish the modular equation of $\lambda(z)$ by employing only the quadratic transformation; this has been done for $n = 3,5,7,11$ and 23 [see the book *Ramanujan's Development of elliptic-function theory* by K Venkatachaliengar, which is to be published by the Madurai Kamraj University. Ramanujan also notices that the most fundamental equation is the modular equation and all other relations can be found by differentiating this equation with respect to q [see e.g. Note book p. 253].

The rich collection of modular relations established by Ramanujan can be found in his note book on several pages; as examples see his letter to Hardy [*Collected Works* p. 353 specially (22)]. The Modular equation for $n=5$ for instance is given in more than a dozen forms.

Perhaps it will take one or two generations to be able to find the beautiful algebraical methods used by Ramanujan in this connection.

Regarding the calculation of singular moduli by Ramanujan in a huge number of cases [*Collected Works* p. 23 et seq] less is known, in spite of the efforts of Watson and others, for $\sqrt{-210}$ Ramanujan has given his method and recently K.G. Ramanathan [*J.I.M.S.* Vol 46,1982 p.104 et seq] has used Kronecker limit formula to derive some of the singular moduli, but this is evidently not the method used by Ramanujan.

In the case when the class number of the corresponding quadratic forms=1 Ramanujan *roughly* indicates his method on p.393 of his note book 2. In one of the recently found scrap pages available at the Trinity College Mathematical Library Ramanujan has worked out the cases for 11,19,27,43,67,163 [3 and 7 are worked earlier in the note book]

$$Q = 1 + 240 \sum_{1}^{\infty} \frac{n^3 q^n}{1-q^n}, \quad R = 1 - 504 \sum \frac{n^5 q^n}{1-q^n}$$

2

Ramanujan and the Additive Theory of Numbers

By
S. Parameswaran

The general problem of the additive theory of numbers may be stated thus. Let $A=\{a_1,a_2,a_3,\ldots\}$ where the elements a_i are special numbers such as primes, natural numbers, squares, cubes etc. Each representation of an arbitrary positive integer n as a sum of the elements of A is called a *partition* of n. Now consider all possible partitions of n in the form

$$n=a_{i_1}+a_{i_2}+\ldots+a_{i_s}$$

where s may be fixed or unrestricted, the a's may or may not be necessarily different and the order may or may not be relevant, according to the particular problem considered. Let us denote by r (n), the number of such representations of n. Then what can we say of r (n) ? For example, is r(n) always positive ? Is there always, at any rate, one representation of every n?

Some of the famous examples of additive number theory are given below :

1. **GOLDBACH Conjecture:** Every even positive integer $n>4$ is the sum of two odd primes i.e., if r(n) is the number of solution of the equation

$$n = P_1 + P_2$$

where n is an even integer greater than four and P_i are odd primes, than $r(n) \geq 1$.

This conjecture dates back to 1742 and is undecided to this day.

II. **Waring's Problem** : To determine, whether, for a given positive integer k, there is an integer s (depending only on k) such that the equation

$$n = a_1^k + a_2^k + \ldots + a_s^k$$

where a_i are non-negative integers, has solutions for every positive integer n.

The problem is named after the English mathematician E. Waring who stated in 1770 (without proof and with limited numerical evidence) that every positive integer n is the sum of 4 squares, of 9 cubes, of 19 fourth powers etc. In this example the partition function r(n) is the number of solutions of the equation given above and the problem is to decide if there exists an s such that $r(n) \geq 1$, for all n. If s exists for a given k, then there is a least value of s and this is denoted by g(k). Lagrange proved that $g(2) = 4$, later in the same year (1770). The exact value of g(k) for every positive integral k, has now been established–216 years after the problem was first enunciated.

UNRESTRICTED PARTITIONS

One of the most fundamental problems in additive number-theory is that of unrestricted partitions.

We take first, the case in which A is the $\{1,2.3,\ldots\}$ of all positive integers, the number of summands is unrestricted, repetitions are allowed and the order of the summands is irrelevant. This is the problem of 'unrestricted partitions'.

Here a partition of a positive integer n, is a representation of n as the sum of any number of positive integral parts. Thus

$$4 = 3+1 = 2+2 \cdot 2+1+1 = 1+1+1+1$$

and

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$$

Denoting by p(n), the number of partitions of n, we have $p(4)=5$ and $p(5)=7$. (We have, of course, to count the representation by one part only) Note that $p(1)=1$.

III. Congruence Properties of p(n):

Very little is known about the arithmetical properties of p(n); we do not know, for example, when p(n) is odd or even. Ramanujan was the first mathematician to discover any such properties and his theorems were discovered, in the first instance, by observation. Mac Mahon had calculated a table of p(n) for the first 200 values of n and Ramanujan observed that the table indicated certain simple congruence properties of p(n). In particular, the numbers of partitions of numbers $5m+4$, $7m+5$ and $11m+6$ are divisible by 5,7 and 11 respectively :

$$\text{i.e, } p(5m+4)=0 \pmod 5$$
$$p(7m+5)=0 \pmod 7$$
$$p(11m+6)=0 \pmod{11}$$

Thus $p(4)=5$ and $p(5)=7$.

Ramanujan not only proved these three congruences-some by more methods than one-(in 1919–1920) but also went a good deal further. He proved congruence with moduli $5^2, 7^2, 11^2$ such as

$$p(25m+24) \equiv 0 \pmod{5^2}$$
$$p(49m+47) \equiv 0 \pmod{7^2}$$
$$a(629m+116) \equiv 0 \pmod{11^2}$$

In connection with these discoveries Ramanujan also stated without proof two remarkable identities.

$$\sum_{m=0}^{\infty} p(5m+4)\, x^m = 5\,\frac{\{f(x^5)\}^5}{\{f(x)\}^6},$$

and

$$\sum_{m=0}^{\infty} p(7m+5)\, x^m = 7\,\frac{\{f(x^7)\}^3}{\{f(x)\}^4} + 49x\,\frac{\{f(x^7)\}^7}{\{f(x)\}^8},$$

where $f(x)=\prod_{n=1}^{\infty}(1-x^n)$.

Since the functions on the right of the two identities have power-series expensions with integral coefficients, these indentities make

the congruences with moduli 5 and 7 intuitive and also provide proofs of the congruences with moduli 5^2 and 7^2.

Ramanujan never published a complete proof of the two identities given above; these identities were proved later by H.B.C. Darling (in 1921), L.J. Mordell (in 1922), H. Rademacher and H.S. Zuckerman (1939) and others.

Besides Ramanujan put forward a general conjecture :—if

$$\delta = 5^a\ 7^b\ 11^c$$

and $24\lambda \equiv 1 \pmod{\delta}$,

then $$p(m\delta + \lambda) \equiv 0 \pmod{\delta}$$

for every m. It would be sufficient to prove the congruence for the special moduli $5^a\ 7^b$ and 11^c and the general congruence would follow as a corollary.

This conjecture has led to a good deal of work. W. Kreemar (in 1933) proved the congruence

$$p(125m + 99) \equiv 0 \pmod{5^3}$$

and G.N. Watson, the congruence for general 5^a . But H. Gupta in extending. Mac Mahon's calculations of p (n) upto n = 300, found that

$$p(243) = 133, 976, 259, 344, 888,$$

a number not divisible by 7^3; and S. Chowla (in 1934) observed that, since

$$24.243 \equiv 1 \pmod{7^3},$$

this contradicts Ramanujan's conjecture for 7^3. The conjecture for 7^b has therefore to be modified and G.N. Watson (in 1938) came out with an appropriate modification, by proving that $p(n) \equiv 0 \pmod{7^b}$ if $b > 1$ and $24n \equiv 1 \pmod{7^{2b-2}}$.

D.H. Lehmer has (in 1936 and 1938) verified the conjecture. in special cases, for 11^3 and 11^4. Joseph Lehner (in 1950) established the congruence

The final result is contained in the main theorem which runs as follows :—

Suppose that

$$\phi_q(n)=\frac{\sqrt{q}}{2\pi\sqrt{2}}\ \frac{d}{dn}\left(\frac{e^{C\lambda_n/q}}{\lambda_n}\right),$$

where $C=\pi\sqrt{(\frac{2}{3})}$ and $\lambda_n=\sqrt{(n-\frac{1}{24})}$, for all positive integral values of q; that p is a positive integer less than and prime to q; that $\omega_{p,q}$ is a 24_q-th root of unity, defined when p is odd by the formula

$$\omega_{p,q}=\left(\frac{-q}{p}\right)\exp\left[-\left\{\frac{1}{4}(2-pq-p)+\frac{1}{12}\left(q-\frac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right],$$

and when q is odd by the formula

$$\omega_{p,q}=\left(\frac{-p}{q}\right)\exp\left[-\left\{\frac{1}{4}(q-1)+\frac{1}{12}\left(q-\frac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right),$$

where (a/b) is the symbol of Legendre and Jacobi, and p′ is any positive integer such that $1+pp'$ is divisible by q; that

$$A_q(n)=\Sigma_p\ \omega_{p,q}\,e^{-2np\pi i/q};$$

and that α is any positive constant, and ν the integral part of $\alpha\sqrt{n}$.

Then

$$p(n)\times\sum_1^{\nu} A_q\,\phi_q+0\,(n^{-1/4}),\ \ldots. \qquad (I)$$

so that p (n) is, for all sufficiently large values of n, the integer nearest to

$$\sum_1^{\nu} A_q\,\phi_q.$$

Prof. J.E. Littlewood writes : "The reader does not need be told that this is a very astonishing theorem .... The story of the theorem is a romantic one. (To do it justice Imust infringe a

$$p(1331m+721)=0 \pmod{11^3},\ m=0,\ 1,\ 2,\ \ldots.$$

H.S. Zuckerman, Morris Newman, 0. Kolberg and others have been able to establish a number of congruences with moduli 13, 17, 19, 23 etc., but these congruences are not so simple as those given by Ramanujan, e.g., a congruence proved by M. Newman is

$$p\left(84n^2-\frac{1}{24}(n^2-1)\right)=0 \pmod{13},\ (n,6)=1.$$

IV. Howlarge is p (n) for Large n ?

It is very remarkable, if we consider how much has been written about approximate or asymptotic values of arithmetical functions, that this question should never have been asked before 1917, when Ramanujan and Hardy published a memoir containing a solution of the problem.

It is easy to find the generating function of p (n); this is the function

$$f(x)=\prod_{r=1}^{\infty}(1-x^r)^{-1}\sum_{n=0}^{\infty}p(n)\,x^n,\ \text{where } p(o)=1.$$

First they show that

$$e^{A\sqrt{n}}<p(n)<a^{B\sqrt{n}} \quad \text{or} \quad A\sqrt{n}<\log p(n)<B\sqrt{n},$$

where A and B are positive constants, for all sufficiently large values of n. Next they prove that

$$\log p(n)\sim C\sqrt{n},\ \text{where } C=\frac{2\pi}{\sqrt{6}};$$

in other words that

$$p(n)=\exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}(1+\epsilon)\right\}$$

where ϵ is small when n islarge.

Then they replace the above formula, by the much more precise one

$$p(n)\sim\frac{1}{4n\sqrt{3}}\exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}\right\}$$

little the rules about collaboration. I therefore add that Prof. Hardy confirms and permits my statements of bare fact). One of Ramanujan's Indian conjectures was that the first term of (I) was a very good approximation to p (n); this was established without great difficulty. At this stage the $(n-\frac{1}{24})$ was represented by a plain n—the distinction is irrelevant. From this point the real attack begins. The next step in development, not a very great one, was to treat (I) as an 'asymptotic' series, of which a fixed number of terms (e.g. $\nu=4$) were to be taken, the error being of the order of the next term. But from now to the very end Ramanujan always insisted that much more was true than had been established: "there must be a formula with error 0 (1)". This was his most important contribution; it was both absolutely essential and most extraordinary. A severe numerical test was made, which elicited the astonishing facts about p (100) and p (200). Then ν was made a function of n; this was a very great step, and involved new and deep function-theory methods that Ramanujan could not have discovered by himself. The complete theorem thus emerged.

'But the solution of the final difficulty was probably impossible without one more contribution from Ramanujan, this time a perfectly characteristic one. As if its analytical difficulties were not enough, the theorem was entrenched also behind almost impregnable defences of a purely formal kind. The form of the function ϕ_q (n) is a kind of indivisible unity; among many asymptotically equivalent forms it is essential to select exactly the right one. Unless this is done at the outset, and the $-\frac{1}{24}$ (to say nothing of the $\frac{d}{dn}$) is an extra-ordinary stroke of formal genius, the complete result can never come into the picture at all. There is, indeed, a touch of real mystery. If only we *knew* there was a formula with error 0 (1), we might be forced, by slow stages, to the correct form of ϕ_q. But why was Ramanujan so certain there was one ? *Theoretical* insight, to be the explanation, had to be of an order hardly to be credited. Yet it is hard to see what numerical instances could have been available to suggest so strong a result. And unless the form of ϕ_q was known already, no numerical

evidence could suggest anything of the kind-there seems no escape, at least, from the conclusion that the discovery of the correct form was a single stroke of insight. We owe the theorem to a singularly happy collaboration of two men, (Hardy and Ramanujan) of quite unlike gifts, in which each contributed the best, most characteristic, and the most fortunate work that was in him. Ramanujan's genius did have this one opportunity worthy of it."

Since p (n) is an integer, the final formula could in principle be used to calculate p (n) *exactly* for large n, and it was fairly easy to decide with moral certainty when enough terms have been taken in any case. But a numerical estimation of the constant in $0\ (n^{-1/4})$, which would have made the conclusion completely rigorous, was not undertaken.

The gap was filled by H. Rademacher in 1937. Here is Prof. G.H. Hardy's account of it : "Rademacher (trying at first merely to simplify our analysis) was led to make a very fortunate formal change. Ramanujan and I worked, not exactly with the function

$$\phi\ (n) = \frac{1}{2\pi\sqrt{2}}\ \frac{d}{dn}\left(\frac{e^{k\lambda_n}}{\lambda_n}\right),$$

but with the "nearly equivalent" function

$$\frac{1}{\pi\sqrt{2}}\ \frac{d}{dn}\left(\frac{\cosh K\lambda_n - 1}{\lambda_n}\right)$$

(after discarding the less important parts of the function). Rademacher works with

$$\psi\ (n) = \frac{1}{\pi\sqrt{2}}\ \frac{d}{dn}\left(\frac{\sinh K\lambda_n}{\lambda_n}\right),$$

which is also "nearly equivalent"; and this apparently slight change has a very important effect, since it leads to an *identity* for p (n). We have

$$\frac{d}{dn}\left(\frac{1}{\lambda_n}\sinh\frac{K\lambda_n}{q}\right) = \frac{d}{dn}\left(\frac{K}{q} + \frac{K^3\ (n-\frac{1}{24})}{6q^3} + \ldots\right) = 0\ \left(\frac{1}{q^3}\right)$$

for fixed n and large q. Hence the function

$$\phi_q(n) = \frac{q^{\frac{1}{2}}}{\pi\sqrt{2}} \frac{d}{dn}\left\{\frac{\sinh(K\lambda_n/q)}{\lambda_n}\right\}$$

behaves for large q like a multiple of $q^{\frac{1}{2}}.q^{-3} = -q^{-\frac{5}{2}}$; and

$$|L_q(n)| = \left|\sum_p \omega_{p,q}\, e^{-2np\pi i/q}\right| \leqslant q,$$

so that

$$\sum_q L_q(n)\, \phi_q(n)$$

is convergent. The series $\Sigma L_q(n)\,\phi_q(n)$ is not convergent; this question Ramanujan and I left doubtful, but it has been settled since by Lehmer (in 1937) Rademacher proved that

$$p(n) = \sum_{q=1}^{\infty} L_q(n)\, \phi_q(n),$$

and that the remainder after Q terms is less than

$$C\, Q^{-\frac{1}{2}} + D\left(\frac{Q}{n}\right)^{\frac{1}{2}} \sinh\frac{Kn^{\frac{1}{2}}}{Q},$$

where C and D are constants for which he found definite values. The remainder is of order $n^{-1/4}$ when Q is of order $n^{1/2}$, as in our older work."

Some Generalizations

1. In 1925, K. Knopp proved the following theorem : "If a_n is an integer and $1 \leqslant a_1 < a_2 < a_3 < \ldots$, $a_n \to \infty$, and n (z), the number of a_n's which do not exceed z, satisfies

$$n(z) \sim k.z^{\alpha} \qquad (k>0,\ \alpha>0)$$

and if

$$f(x) = \prod (1-x^{a_i})^{-1} = \sum p(n).\, x^n,$$

(so that p (n) is the number of partitions of n into a's), then

$$\log p(n) \sim \frac{1+\alpha}{\alpha} \cdot \beta^{\frac{1}{1+\alpha}} \cdot n^{\frac{\alpha}{\alpha+1}} \text{ where } \beta = k.\ \alpha.\ \Gamma(1+\alpha).\ \xi(1+\alpha).$$

II. In 1942 P. Erdös established the following results :—

(i) Let $a_1 < a_2 < \ldots$ be an infinite sequence of integers of density α, such that the a's have no common factor. Denote by $p'(n)$ the number of partitions of n into the a's. Then

$$\log p'(n) \sim c\ (\alpha.n)^{1/2} \text{ where } c = \pi\ (2/3)^{1/2}$$

(ii) Let $a_1 < a_2 < \ldots$ be an infinite sequence of integers of density α, such that every sufficiently large m can be expressed as the sum of different a's. Then

$$\log p'(n) \sim c.\ \left(\frac{\alpha}{2} \cdot n\right)^{1/2}$$ where $p'(n)$ denotes the number of partitions of n into different a's.

A Further Generalization

Let us define '*a slowly oscillating function*' and '*pairs of conjugate slowly oscillating functions*' first.

A positive-valued function L (x) defined for $x \geqslant a$, where a is some positive number, is called *slowly oscillating* if it is measurabl and if for every $c > 0$ we have

$$\frac{L(cx)}{L(x)} \longrightarrow 1 \text{ as } x \to \infty$$

This notion, with restriction to continuous functions, is due to Karamata (1930).

If L (x) is slowly oscillating, then there exists a slowly oscillating function L^* such that

$$L^*(xL(x)).\ L(x) \to 1 \qquad (x \to \infty)$$
$$L(xL^*(x)).\ L^*(x) \to 1 \qquad (x \to \infty)$$

L^* is asymptotically uniquely determined and the relation $L \to L^*$ is involutory in the sense that L and $(L^*)^*$ are asymptotically equivalent

L^* is called the conjugate of L. according to N.G. de Bruijn (1959).

Let A be a denumerable set of distinct positive numbers without finite limit point. Then the elements of the set can be arranged in a sequence $a_1<a_2<a_3....$, where a_k tends to infinity with k. Let $0=b_0<b_1<b_2<....$ be the elements of the additive semi-group generated by A. Consider the weighted partition function $p(b_m)$ defined by the generating function :

$$f(x)=\prod_{a_k \epsilon A}(1-x^{a_k})^{-\psi(k)}=\sum p(b_m).x^{b_m}$$

where $\psi(k)$ is a function on the positive integers into the non-negative reals such that the sum and product are both convergent for all positive x less than one. (Actually convergence of either the sum or the product implies convergence of the other). Here

$$p(b_i)=\sum_{m_1a_1+m_2a_2+....=b_i}\prod_{k=1}(\psi(k)+m_k-1)$$

where the summation is taken over non-negative integers m_k and is infinite. (The k-th factor in the product is 1 for all k for which $a_k>b_i$, since m_k must be zero). When $\psi(k)=9$ for all positive integers k, $p(b_i)$ is just the number of unrestricted partitions of b_i into parts taken from the set A.

Letting $n(u)=\sum_{a_k\leqslant u}\psi(k)$ and $p(u)=\sum_{b_i\leqslant u}p(b_i)$, E.E. Kohlbecker (in 1958) proved that an asymptotic relation of the form

$n(u)\sim u^{\alpha}.L(u)$ as $u\to+\infty$, where α is a positive constant and L is a slowly oscillating function in the sense of Karamata (given above), is equivalent to an asymptotic relation of the form

$$\log p(u)\sim u^{\frac{\alpha}{\alpha+1}}L_1(u) \text{ as } u\to+\infty,$$

where L_1 is a slowly oscillating function related to L by a certain implict formula which can be solved for L_1 in the cases usually encountered. The special case of this where L (u) is a constant had been given earlier by Knopp and Erdos (mentioned in the previous section). When L (u) is a power of log u, the deduction in one direction (from n (u) to log p (u) had been given by N.A. Brigham (in 1950).

In all the cases mentioned above a is assumed to be a positive quantity. S. Parameswaran in his paper (of 1961) considers the case where $a = 0$. It is shown that an asymptotic relation of the form $n(u) \sim L(u)$ as $u \to +\infty$, where $L(u)$ is a slowly oscillating function, implies an asymptotic relation of the form $\log p(u) \sim [M^*(u)]^{-1}$ as $u \to +\infty$. Here $M^*(x)$ is a slowly oscillating function related to

$$M(x) = \int_{x_0}^{x} \frac{L(u)}{u}\, du,$$

and hence to $L(x)$ ultimately, by means of an implicit formula, which can be solved for M^* in the cases usually met with; more specifically, M^* is the conjugate slowly oscillating function to M in the sense of N.Q. de Bruijn (defined at the beginning of this section). The deduction in the reverse direction has been made under further hypothesis on $L(u)$, namely 'provided $L(u) \sim J(u)$ as $u \to +\infty$, where $J(u)$ is differentiable and $u(J'(u)/J(u))$ is non-increasing'.

About the main theorem of this section, Prof. L.J. Mordell says that its proof depends "upon Cauchy's integral for a function of a complex variable, a subject of which Ramanujan knew practically nothing. Of course, the first suggestions as to the possibility of such results and some indication of their form were due to Ramanujan, and no doubt he was full of suggestions as to the form the final results should take. There would obviously have been no such formula without Ramanujan, and one can easily understand Hardy's gratitude for having come in contact with Ramanujan."

3

Identities of Rogers-Ramanujan Type

by A. Verma

The attention that the following two beautiful identities of the genius Indian mathematician Ramanujan

$$(1.1)\qquad \sum_{n>0} \frac{q^{n^2}}{[q;q]_n} = \prod_{\substack{n=0 \\ n\neq 0,\pm 2 \pmod 5}}^{\infty} (1-q^n)^{-1}$$

$$(1.2)\qquad \sum_{n>0} \frac{q^{n^2+n}}{[q;q]_n} \prod_{n\neq 0,\pm 1 \pmod 5}^{\infty} (1-q^n)^{-1}$$

where $[q;q]_0 = 1$, $[a;q]_n = (1-a)(1-aq)\ldots(1-aq^{n-1}), |q|<1$

have received is unparalleled by the attention paid to any particular identity in analysis, though Ramanujan himself did not know how to prove them. Analytical identities analogous to them and their generalizations in diverse directions have been the subject matter of investigation of numerous brilliant analysts for the past seven decades. Professor G.H. Hardy to whom they were communicated, along with the other results of Ramanujan was baffled by these identities. He tried to prove them, but in vain. They were communicated by Hardy to P.A. Mac Mahon for comments. Though Mac Mahon could not also prove them yet he observed that they were equivalent to the following partition theoretic identities:

(1.1)′ The first result is equivalent to "The number of partitions of n in which the minimal difference between the summands

is at least 2,equals the number of partitions of n into parts of the form $5m+1$ and $5m+4$".

(1.2)′ Similarly, the second identity is equivalent to "The number of partitions of n into parts greater than 1 in which the minimal difference between the summands is 2, equals to the number of partitions of n into parts of the form $5m+2$ and $5m+3$".

Mac Mahon also pointed out to Hardy that he had verified these results for all integers $\leqslant 69$.

Ramanujan while going through some of the old volumes of the *Proceedings of the London Mathematical Society* came across the little known works of L.J. Rogers [62–64] which contained analytical proofs of not only these identities but also contained a host of other similar identities for the modulii 5,7,10,14,15,20 and 21. Hardy arranged to publish in 1919 a joint paper of Rogers and Ramanujan [66] which contained analytical proofs of these identities. It may be pointed out that in 1917, I.J. Schur [67] rediscovered these identities independently. The amount of interest these identities have generated in number theorists and analysts can be imagined by the fact that at least six different proofs of these identities were published in the next decade. One of the proofs of these identities given in Rogers–Ramanujan's joint paper [66] is based on proving the formula

$$\sum_{n\geqslant 0} (-1)^n a^{2n} q^{\frac{1}{2}(5n-2)} \frac{(a-aq^{2n})}{1-a} \frac{[a;q]_n}{[q;q]_n}$$

$$= [q;q]_\infty \sum_{n\geqslant 0} \frac{a^n q^{n^2}}{[q;q]_n}, \quad |q|<1$$

which for $a=1$ and $a=q$ reduce to (1.1) and (1.2) respectively on summing the left hand side by the well-known triple product identity of Jacobi

$$(1.3)\quad \sum_{n=-\infty}^{\infty} (-)^n z^n q^{an^2} = \prod_{n=1}^{\infty} [(1-q^{2an})(1-zq^{(2n-1)a})\left(1-\frac{q^{(2n-1)a}}{z}\right)]$$

$$(z\neq 0, |q|<1)$$

(see Andrews [4] for an elegant proof of (1.3).

$$ {}_{r+3}\phi_{r+2}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a_1, a_2, \ldots, a_r; q; z \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{a_1}, \frac{aq}{a_2}, \ldots, \frac{aq}{a_r} \end{matrix}\right] $$

would be abbreviated as ${}_{r+3}W_{r+2}[a;a_1,a_2,\ldots,a_r;q;z]$, and is referred to as a very well-poised basic hypergeometric series.

In this terminology Watson's q-analogue of Whipple's transformation (1.4) is a relation connecting a very well-poised terminating ${}_8W_7$ with a Saalschutzian ${}_4\phi_3$. Hardy [41] in 1940 had given a proof of Rogers-Ramanujan identities (1.1) and (1.2) by combining the Rogers'method given in [63] with a summation formula which is a limiting case of Jackson's [52] summation formula for terminating very well-poised balanced ${}_8W_7$. Later on in 1943 Dyson [46] used a similar method for obtaining identities of Rogers-Ramanujan type for the modulus 7. (In fact Dyson [46] pointed out that these identities were rediscovered by A. Selberg [69] and can be found in Rogers original paper [63]). Although, Dyson did not use the Jackson's summation formula for ${}_8W_7$ but yet his formulae for α_n,β_n are actually derivable from the Jackson's [52] summation formula. To obtain Dyson's formula for γ_n by a direct limiting process it appears necessary to use Watson's transformation (1.4). Bailey [36] used the Rogers technique together with known summation formilae of basic hypergeometric series to obtain identities of Rogers-Ramanujan type. The most interesting amongst them were the identities for modulii 8,9,10,27.

At that point of time, it was clear that the transformations of basic hypergeometric series would play significant role in the development of analytical identities of Rogers-Ramanujan type. Indeed, in 1947 Bailey [37] showed that using the simple general identity

$$ \sum_{n=0}^{\infty} \alpha_n \gamma_n \sum_{n=0}^{\infty} \beta_n \delta_n $$

where $\beta_n = \sum_{n=0}^{n} u_{n-r} v_{n+r} \alpha_r$

$$ \gamma_n = \sum_{r=n}^{\infty} u_{r-n} v_{r+n} \delta_r $$

An elegant proof of the identity (1.3) was given by G.N. Watson [82] in 1929. He obtained it as a limiting case of his q- analogue of whipple's transformation.

$$(1.4)\quad {}_8W_7\left[a;\ c,\ d,\ e,\ f,\ g;\ q;\ \frac{a^2q^2}{cdefg}\right]=\frac{\left[aq,\ \frac{aq}{fg},\ \frac{aq}{eg},\ \frac{aq}{ef}\ ;\ q\right]_\infty}{\left[\frac{aq}{e},\ \frac{aq}{f},\ \frac{aq}{g},\ \frac{aq}{efg}\ ;\ q\right]_\infty}\cdot$$

$$\cdot 4\phi_3\left[\begin{matrix}\frac{aq}{cd},\ e,\ f,\ g;\ q;\ q\\ \frac{efg}{a},\ \frac{aq}{c},\ \frac{aq}{d}\end{matrix}\right]$$

where e, f or g is of the form q^{-N} (N is a non-negative integer) by letting c,d,e,g$\to\infty$. The generalised basic hypergeometric series ${}_r\phi_{r-k+1}$ is defined as

$${}_r\phi_{n-r+1}\left[\begin{matrix}a_1,\ a_2,\ldots,\ a_r;\ q;\ z\\ b_1,\ b_2,\ldots,b_{r-k+1}\end{matrix}\right]$$

$$=\sum_{n>0}\frac{[a_1,\ a_2\ldots.\ a_r;\ q]_n\ (-1)^{kn}\ q^{\frac{kn}{2}(n-1)}\ z^n}{[q,\ b_1,\ b_2,\ldots,\ b_{r-k+1}\ ;\ q]_n},$$

$q<1$, where $[a_1;q]_n[a_2;q]_n\ldots[a_r;q]_n$ has been abbreviated as $[a_1,a_2,\ldots,a_r;q]_n$. The above series converges for all values of z for $k>0$ for $k<0$ converges only for $z=0$ and for $k=0$ it converges in side the unit disc $|z|<1$, The generalised basic hypergeometric series ${}_r\phi_{r-1}\left[\begin{matrix}a_1a_2,\ldots,a_r;q;z\\ b_1,b_2,\ldots,b_{r-1}\end{matrix}\right]$ is called well poised if $qa_1=a_2b_1=\ldots=a_rb_{r-1}$ and if any one of these equalities break down then it would be called a nearly-poised basic hypergeometric series and if $qa_1a_2\ldots a_r=b_1b_2\ldots b_{r-1}$ we call ${}_r\phi_{r-1}$ Saalschützian or balanced basic hypergeometric series. The special well poised basic hypergeometric series

obtained transformations of basic hypergeometric series by giving suitable values to $\langle u_n \rangle$, $\langle v_n \rangle$, $\langle a_n \rangle$ and $\langle \delta_n \rangle$ and using the various summation formulae of basic hypergeometric series from which analytical identities of Rogers-Ramanujan type were obtained. L.J. Slater [76,77] used the above method of Bailey [37] to obtain as many as 130 identities of Rogers-Ramanujan type on the modulii 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 27, 30, 32, 42, 48 54, 64.

On the partition theoretic front I.J. Schür [68] in 1926 had proved the following result which is very similar to Rogers-Ramanujan identities (1.1)′ and (1.2)′ :

(1.3)′ The number of partitions of n of the form $n=b_1+b_2+\ldots+b_s$ where $b_i-b_{i+1}\geqslant 3$ and $b_i-b_{i+1}>3$ if $b_i/3$ is equal to the number of partitions of n into parts of the form $6m+1$ and $6m+5$.

However, all the hopes of generalization in this direction came to an end in 1948 when H.L. Alder [1] obtained the following three inexistence theorems :

Let S be any fixed set of positive integers and $\lambda_{d,m}(n)$ the number of partitions of n into parts differing by at least d with each part being greater than or equal to m.

(1.4)′ $\lambda_{d,m}(n)$ is not always equal to the partitions of n into parts taken from S if $d>2$.

(1.5)′ $\lambda_{d,m}(n)$ is not equal to the number of partitions of n into discount parts taken from S if $d>1$.

(1.6)′ The number of partitions of n into parts differing by at least d and where no consecutive multiples of d appear is not equal to the number of partitions of n into parts taken from S if $d>3$.

But in 1954, H.L. Alder [2] proved that there exist polynomials $G_{k,n}(q)$ such that

$$(1.5)\qquad \prod_{\substack{n=1\\ n\neq 0,\pm k \pmod{2k+1}}}^{\infty}(1-q^n)^{-1} = \sum_{n=0}^{\infty}\frac{G_{k,n}(q)}{[q;q]_n},$$

$$(1.6)\qquad \prod_{\substack{n=1\\ n\neq 0,\pm 1 \pmod{2k+1}}}^{\infty}(1-q^n)^{-1} = \sum_{n=0}^{\infty}\frac{G_{k,n}(q)}{[q;q]_n}q^n$$

He himself showed that $G_{2,n}(q)=q^{n^2}$ and so for $k=2$, they reduce to (1.2) and (1.2) respectively. However, the left hand side of (1.5) is the generating function for the number of partitions into parts which are not congruent to $0, \pm k \bmod (2k+1)$. However, the left hand side of (1.6) represents the number of partitions into parts not congruent to $0, \pm 1 \bmod (2k+1)$. Unlike the Rogers-Ramanujan identities the right hand sides of Alder generalization had no combintorial interpretation for $k \geqslant 3$. Alder's proof of (1.5)-(1.6) was based on certain auxilary functions introduced many years ago by Selberg [69]. Analytic generalization of Alder's identities were given by V.N. Singh [73-75] by using transformations of basic hypergeometric series developed by D.B. Sears [70-72] Singh obtained an explicit expression for $G_{k,n}(q)$. Carlitz [41] also gave further properties of these polynomials. Andrews [23] succeeded in giving a partition theoretic interpretation of (1.5) and (1.6). Andrews ingenuity lies in his observation that (1.1) and (1.2) also have the following alternative partition theoretic interpretations which are completely different from (1.1)′ and (1.2)′.

(1.7) The number of partitions of n in which each part is at least as large as the number of parts equals the number of partitions of n into parts congruent to $\pm 1 \pmod 5$.

(1.8)′ The number of partitions of n in which each part is larger than the number of parts equals the number of partitions of n into parts congruent to $\pm 2 \pmod 5$.

In this sequel it may be mentioned that the case $d=2$ of (1.6)′ was treated by H. Gollnitz [48] in 1960 and later-on, independently by B. Gordon [50] in 1965. They proved the following two identities.

(1.9)′ The number of partitions of any positive integer n into parts $= -1, 4, 7 \pmod 8$ is equal to the number of partitions of the form $n=b_1+b_2+\ldots.+b_s$ where $b_i - b_{i+1} \geqslant 2$ and $b_i - b_{i+1} > 2$ if b_i is even.

(1.10)′ The number of partitions of any positive integer n into parts $= 3, 4, 5 \pmod 8$ is equal to the number of partitions of the form $n=b_1+b_2+\ldots.+b_s$ satisfying $b_i \geqslant 3$ in addition to the inequalities in (1.9)′.

Another form of generalization of Rogers-Ramanujan identities was given by Gordon [49] in 1961. (for further identities of this nature see Andrews [7, 8, 9, 10]. He proved the following :

(1.11)′ Let a and k be integers $o<a\leqslant k$. Let $A_{k,a}(n)$ denote the number of partitions of n in to parts $= o, \pm a \pmod{2k+1}$. Let $B_k; a(n)$ denote the partitions of n of the form $n = \sum_{i=1}^{\infty} i.f_i$ (where f_i is the number of times the part i appears in the partition) with $f_i \leqslant a-1$ and for all $i \geqslant 1$, $f_i - f_{i+1} \leqslant k-1$ then $A_{k,a}(n) = B_{k,a}(n)$.

Gordon's generalization (1.11)' reduces to (1.1)' for $k=2=a$ whereas for $k=2$, $a=1$ it reduce to (1.2)′. Andrews [11] obtained the following generalization of (1.9)' and (1.10)' (exactly in the same manner in which Gordon's identity (1.11)' generalizes (1.1)' and (1.2)') :

(1.12)′ Let a and k be integers with $o<a\leqslant k$ and $C_{k,a}(n)$ be the number of partitions of n into parts which are neither $= 2 \pmod 4$ nor $= o, \pm (2a-1) \pmod{4k}$. Let $D_{k,a}(n)$ denote the number of partitions of n of the form $n = \sum_{i=1}^{\infty} i.f_i$ with $f_i + f_2 \leqslant a-1$ and for all $i \geqslant 1$, $f_{2i-1} \leqslant 1$ and $f_{2i} + f_{2i+1} + f_{2i+2} \leqslant k-1$ then $G_{k,a}(n) = D_{k,a}(n)$. Clearly (1.12)′ reduces to (1.9)′ for $k = a = 2$ whereas for $k=2$, $a=1$ it reduces to (1.10)′. Andrews [21] had obtained very elegant partition theoretic identities which contain as special cases not only Gordon's generalization (1.11)′ of Rogers-Ramanujan identities and his generalization. (1.12)′ of Gollnitz Gordon identities but also provide generalization of a number of other known partition identities (see also [12, 14, 17]).

2. Multidimensional identities. In another important paper in 1974 Andrews 24] obtained the following k-dimensional analogue of Rogers-Ramanujan identities (1.1) and (1.2)

$$(2.1)\quad \sum_{n_1, n_2, \ldots, n_{k-1} \geqslant o} \frac{q^{N_1^2 + N_2^2 + \ldots N_{k-1}^2 + N_i + N_{i+1} + \ldots N_{k-1}}}{[q;q]_{n_1}\,[q;q]_{n_2} \ldots [q;q]_{n_{k-1}}}$$

$$= \prod_{n \neq o, \pm i \pmod{2k+1}} (1-q^n)^{-1},$$

Where $N_j = n_j + n_{j+1} + \ldots + n_{k-1}$. In fact (2.1) for $k=i=2$ reduces to (1.1) and for $k=2, i=1$ it reduces to (1.2). Further more these $(k-1)$ identities are closely related to Gordon's [49] partition theoretic identity [30] :

(2.1)' Let a and k be integers with $o<a\leqslant k$. Let $A_{a,k}(n)$ denote the number of partitions of n into parts $\neq o, \pm a \pmod{2k+1}$. Let $B_{k,a}(n)$ denote the number of partitions of n of the form $n=b_1+b_2+\ldots+b_s$ with $b_i \geqslant b_{i+1}$, $b_j - b_{i-k-1} \geqslant 2$ and with 1 appearing as summand at most $a-1$ times. Then $A_{k,a}(n) = B_{b,a}(n)$.

Yet another result of Andrews [24] is the following k-dimensional identity of Rogers-Ramanujan type deduced as a limiting case of a k-dimensional analogue of Watson's q-analogue of Whipple's transformation (1.4) :

$$\sum_{n_1, n_2, \ldots, n_{k-1} \geqslant o} \frac{[-q;q^2]_{N_1}\, q^{N^2_1 + 2(N^2_2 + N^2_3 + \ldots N^2_{k-1})}}{[q^2;q^2]_{n_1}\, [q^2;q^2]_{n_2} \ldots [q^2;q^2]_{n_{k-1}}}$$

$$= \prod_{\substack{n \neq o, \pm(2k-1) \pmod{4k} \\ n \neq 2 \pmod 4}} (1-q^n)^{-1}$$

which is an analytic counter, part of his partition theoretic identity (1.12)'. Starting with the k-dimensional analogue of transformations of basic hypergeometric series k-dimensional analogues of Rogers-Ramanujan identities for moduls 7 were obtained in [80].

Recently, in 1984 Andrews [29] showed that all the 130 Rogers-Ramanujan identities given in Bailey [37] can be emt edded in an infinite family of identities. For instance the following identity of Rogers [63] involving quintuple products.

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{[q;q]_{2n}} = \prod_{n=o}^{\infty} \left[\frac{(1-q^{10n+10})\,(1-q^{10n+2})\,(1-q^{10n+8})\,(1-q^{10n+6})\,(1-q^{10n+14})}{(1-q^{n+1})} \right.$$

was shown to be imbeded in the infinite family of identities

$$\frac{\sum_{n_k - n_{k-1} > \ldots > n_1 > 0} q^{n_1^2 + n_2^2 + \ldots + n_k^2}}{[q;q]_{n_k n_{k-1}} [q;q]_{n_{k-1} - n_{k-2}} \ldots [q;q]_{n_2 - n_1} [q;q]_{2n_1}} =$$

$$= \pi \, (1 - q^n)^{-1}$$
$$n \neq o, \pm (k+1) \pmod{6k+4}$$
$$n \neq o, \pm (4k+2) \pmod{(12k+8)}$$

The proof of these results was based on finding a k-dimensional analogue of Bailey's transformation corresponding to the choice of the sequences $u_r = \frac{1}{[q;q]_r}$, $v_r = \frac{1}{[q;q]_r}$,

$$\delta_r = \frac{[p_1;q]_r \, [p_2;q]_r \, [\frac{aq}{p_1 p_2};q]_{N-r}}{[q;q]_{N-r} \, [\frac{aq}{p_1};q]_N \, [\frac{aq}{p_2};q]_N} \quad \left(\frac{aq}{p_1 p_2}\right)^r$$

and for arbitrary sequences $\langle \alpha_r \rangle, \langle \beta_r \rangle$ and then choosing $\langle \alpha_r \rangle$ to be one of the Bailey's pairs so that $\langle \beta_n \rangle$ could be evaluated in a closed form.

A closer examination of the identities discussed so far (though a sample section) suggests the more general problem of considering the existence of identities of the type.

$$(2.2)\quad 1 + \sum_{n=1}^{\infty} A_{k,a}(n) \, q^n = \prod_{n \neq o, \pm a \pmod{2k+1}} (1 - q^n)^{-1},$$

where $A_{b,a}(n)$ denotes the number of partitions of n into parts $\neq o, \pm a \pmod{2k+1}$. Non-trivial expressions for the function $A_{k,a}(n)$ for $k = 2, 3, 4$ are known. They correspond to Rogers-Ramanujan identities of modulii 5, 7 and 9 respectively. The following general question was posed in 1969 by a group of number theorists at the summer institute on number theory held in Stony-Brook (U.S.A.) "Are there non-trivial expressions for the function $A_{k,a}(n)$ in (2.2) when $k \geq 5$ in particular $k = 5$".

An affirmative answer to this question was provided for $k = 5$ by Andrews [22]. Andrews [22] arrived at a very important conclusion while examining the list of 130 identities of Rogers-Ramanujan type given in Slater [76, 77]. He observed that

in all these identities the modulii involved in the exponents of the infinite products are always $\leqslant 64$ and always have prime factors from amongst $\{2, 3, 5, 7\}$ i. e. corresponding to k 1, 2, 3. He then argued that because of this the identities related to modulus 11 (i. e. corresponding to k=S) do not fit into the existing general setting of the transformation theory of basic hypergeometric series. As a result of his intuition, he relied upon the technique of q-difference equation to obtain the following elegant transformation [22; Theorem 5.1] which in the notation of basic hypergeometric series would read as

$$(2.3)\quad {}_8\phi_6\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, iq^{-n}, -iq^{-n}, -q^{-n}, q^{-n}, c\\ \sqrt{a}, -\sqrt{a}, -iqa^{1+n}, iq^{1+n}a, -aq^{1+n}, aq^{1+n}\end{matrix};q;\ -aq^{1+4n}\right]$$

$$=\frac{[aq;q]_{4n}}{[a^4q^{4+4n};q^4]_n}\ {}_3\phi_2\left[\begin{matrix}-q^{-2n}, q^{-2n}, 0\\ \frac{q}{a}^{-4n}, \frac{q}{a}^{1-4n}\end{matrix};q^2;q^2\right]$$

Using (2.3) he deduced as special cases non-trivial expressions for the function $A_{5,a}(n)$ thus obtaining identities of Rogers-Ramanujan type related to modulus 11.

In (1982), Jain [55] used the quadratic transformation (obtained by Askey and Wilson [31] and later proved by Jain [54] by a straight forward manupulation of series)

$${}_4\phi_3\left[\begin{matrix}a^2, b^2, c, d\\ ab\sqrt{q}, -ab\sqrt{q}, -cd\end{matrix};q;q\right]={}_4\phi_3\left[\begin{matrix}a^2, b^2, c^2, d^2\\ a^2b^2q, -cd, -cdq\end{matrix};q^2;q^2\right]$$

where a, b, c or d is of the from q^{-N}, N being a non-negative integer, to produce the following transformation of a very well-poised ${}_8W_7$ on base q into Saalschutzian ${}_4\phi_3$ on base q^2 (which is a reminicent of Watsons q-analogue of Whipple's transformation (1.4).

$$(2.4)\quad W\left[a;c,e,-e,-q^{-n},q^{-n};q;\ \frac{a^2q^{+2n}}{ce^2}\right]$$

$$=\frac{[a^2q^2;q^2]_n\left[-\frac{aq}{2};q\right]_n\left[\frac{a^2q^2}{c^2e^2};q^2\right]_n e^{2n}}{\left[\frac{a^2q^2}{c^2};q^2\right]_n\left[\frac{a^2q^2}{e^2};q^2\right]_n[-aq;q]_{2n}}.$$

$$ {}_4\phi_3\left[\begin{matrix}\frac{ce^2}{a^2}q^{-2n-1}, \frac{ce^2}{a^2}q^{-2n}, e^2, q^{-2n}; q^2; q^2\\ \frac{e^2c^2q^{-2n}}{a^2}, \frac{-e^2q^{-2n}}{a}, \frac{-e^2q^{1-2n}}{a}\end{matrix}\right] $$

(2.4) for $c\to 0$ and then setting $e=iq^{-n}$ reduces to (2.3) which yielded Rogers-Ramanujan identities of modulus 11. Whereas (2.4) for $c\to\infty$ yields a transformation which was used by Jain [53] to obtain Rogers-Ramanujan identities related to modulus 13.

It is natural to enquire whether the transformations (2.3) or for that matter (2.4) are isolated results or they are part of some general frame-work of transformations of basic hypergeometric functions. In [80, 81] number of transformations of basic hypergeometric functions on one base to basic hypergeometric functions on another base were developed. In particular, the following transformation connecting very well-poised ${}_{10}W_9$ (on base q) and Saalschützian ${}_5\phi_4$ (on base q^2)

$$ (2.5)\quad {}_{10}W_9\left[a;b,x,-x,y,-y,q^{-n}-q^{-n};q;-\frac{a^2q^{3+2n}}{bxy^2}\right] $$

$$ =\frac{[a^2q^2;q^2]_n\,[\frac{a^2q^2}{x^2y^2};q^2]_n}{[\frac{a^2q^2}{x^2};q^2]_n\,[\frac{a^2q^2}{y^2};q^2]_n}\;{}_5\phi_4\left[\begin{matrix}x^2,y^2,\frac{aq}{b},-\frac{aq^2}{b},q^{-2n};q^2;q^2\\ -aq,-aq^2,\frac{a^2q^2}{b^2},\frac{x^2y^2q^{-2n}}{a^2}\end{matrix}\right] $$

was obtained in [80] (non-terminating version of (2.5) was given in [81] which besides containing (2.4) and (2.3) as special cases was used [80] for obtaining Rogers-Ramanujan identities related to modulii 17 and 19. Rogers-Ramanujan identities related to modulii 22, 26, 33, 34, 39, 51, 59 were also obtained [80, 81].

In this article attention has been confined to only a few directions in which the Rogers-Ramanujan identities have been extended. There are a number of other directions in which the Rogers-Ramanujan identities have been extended, for instance, to polynomial identities which imply Rogers-Ramanujan identities (Bressaud [33] and Andrews [15, 25]), compatiability of q-difference equations [19] and continued fractions [42].

Yet another question that arises in one mind is whether the Rogers-Ramanujan type of partition identities or their analytic counter parts are of any relevance to the rest of mathematics. The answer is surely yes. Indeed the recent rapid advances of Lie algebras starting with I an Macdonald's [61] major break through has produced significant results illustrating the relation between Lie algebras and partition identities (see Lapowsky and Milne [60], Kac [56]. This work has produced a fully Lie algebraic proof of Rogers-Ramanujan identities (1.1) and (1.2).

References

1. H. L. Alder The non-existence of certain identities in the theory of partitions and compositions. *Bull Amer. Math. Soc.* 54 (1948), 712-722.
2. ——— Generalization of the Rogers-Ramanujan identities *Pacific J. Math.* 4 (1954), 161-168.
3. ——— Partition identities from Euler to the present. *Amer. Math. Monthly* 76 (1969), 733-746.
4. G. E. Andrews A simple proof of Jacobi's triple product identity. *Proc. Amer. Math. Soc.* 16 (1965), 333- 34.
5. ——— On basic hypergeometric series. Mook theta functions and partitions (II) *Quart J. Math.* (Oxford) (2) 17 (1966) 132-143.
6. ——— q-identities of Auluck, Carlitz and Rogers. *Duke Math. J.* 33(1966), 575-582
7. ——— An analytic proof of Rogers-Ramanujan-Gorden identities. *Amer. J. Math.* 8? (1966), 844-856.
8. ——— Partition theorems related to Rogers-Ramanujan identities. *J. Combinatorial Theory* 2(1967), 422-430.
9. ——— Some new partition Theorems *J. Combinatorial Theory* 2(1967), 431-436.
10. ——— Partition theorems related to Roger-Ramanujan identities. *Combinatorial Theory* 2(1967), 2(1267), 437-465.

11. ——— A generalization of Gollnitz-Gordon partition theorems *Proc. Amer. Math. Soc.* 18(1967), 945-952.
12. ——— On q-difference equations for certain well-poised basic hypergeometric series. *Quart. J. Math.* (Oxford) (2) 19 (1968), 433-447.
13. ——— A general theorem on partitions with difference conditions. *Amer. J. Math.* 91(1969), 18-24.
14. ——— A generalization of the classical partition theorems. *Trans. Amer. Math. Soc.* 145(1969) 205-221.
15. ——— A partition identity which implies the Rogers-Ramanujan identities. *Scripta Math* 28(1970), 297-305.
16. ——— A new property of partitions with applications to Rogers-Ramanujan identities. *J. Combinatorial Theory* 10(1971), 266-270.
17. ——— Partition identities *Advances in Math.* 9(1972). 10-51.
18. ——— A general theory of identities of Rogers-Ramanujan type *AMS Audio Recording No. 75* and supplementry Manual, Providence 1972.
19. ——— On the Rogers-Ramanujan indentities and partial q-difference equations. *Illinois J. Math.* 16 (1972), 270-275.
20. ——— Analytic generlization of Rogers-Ramanujan identities for odd modulus. *Proc. Nat. Acad. Sc. USA* 71 (1974), 4082-4085.
21. ——— On the general Rogers-Ramanujan theorems. *Memoirs of the Amer. Math. Soc.* No 152 (1974)
22. ——— On Rogers-Ramanujan identities related to modulus 11. *Proce. London Marh. Soc.* (3) 30 (1975), 330-346.
23. ——— On the Alder's polynomials and a new generalization of Rogers-Ramanujan identities. *Trans. Amer. Math. Soc.* 204 (1975), 40-64.
24. ——— Problems and prospects for basic hypergeometric functions, The theory and applications of special functions (R. Askey ed.) Academic Press (1975), 191-224.
25. ——— On identities implying Rogers-Ramanujan identities. *Houston J. Math.* 2 (1976), 289-298.

26. ——— The theory of partitions. Encyclopedia of Mathematics and applications Vol. 2. Addision-wesley, Reading 1977.

27. ——— Partitions and Durfee disection *Amer. J. Math.* 101 (1979), 735-742

28. ——— The Hard hexagon model and Rogers-Ramanujan type identities. *Proc. Nat. Acad. Sc. USA.* 78 (1981), 5290-5292.

29. ——— Multiple series Rogers-Ramanujan type identities *Pacific J. Math.* 114 (1984), 267-283.

30. G. E. Andrews and R. Askey Enumeration of partitions : The role of Eulerian series and q-orthogonal polynomials in Higher combinatorics (M. Aigner ed.) Reidel, Dordrecht 1977.

31, R. Askey and J. Wilson Some basic hypergeometric polynomials that generalize Jacobi polynomials. *Memoir. Amer. Math. Soc.* No. 319 (1985).

32. D. M. Bressoud Analytic and combinatorial generalizations of Rogers-Ramanujan identities. *Memoirs Amer. Math. Soc.* No. 227 (1980).

33. ——— Some identities for terminating q-series *Proc. Comb. Phil. Soc.* 89 (1981), 211-223.

34. W. N. Bailey Series of hypergeometric type which are infinite in both directions. *Quart. J. Math.* (Oxford) 7 (1936), 105-115.

35. ——— A note on certain q-identities. *Quart. J. Math.* (Oxford) 12 (1941), 173-175.

36. ——— Some identities in Combinatory Theory *Proc. Roudon Math. Soc.* (2) 49 (1947), 421-435.

37. ——— Identites of Rogers-Ramanujan type *Proc. London Math. Soc.* (2) 50 (1949), 1-10.

38. ——— On the simplification of some identities of Rogers-Ramanujan type. *Proc. London Math. Soc.* (3) 1 (1951), 217-221.

39. L. Carlitz A note on Rogers-Ramanujan identities. *Math. Nach.* 17 (1958), 23-26.

40. ——— Some formulas related to Rogers-Ramanujan identities. *Annali di Math.* (4) 47 (1959), 243-252.

41. ——— Note on Alder's polynomials. *Pacific J. Math.* 10 (1960), 517-519.

42. ——— Note on some continued fractions of Rogers-Ramanujan type. *Duke Math. J.* 32 (1965), 713-720.

43. ——— A note on Rogers-Ramanujan identities. *Duke Math. J.* 35 (1968), 830-842.

44. ——— Some identities in combinatorial analysis *Duke Math. J.* 38 (1971), 51-56.

45. J. M. Dobbie A simple proof of Rogers-Ramanujan identities *Quart. J. Math.* (Oxford) (2) 13 (1962), 31-34.

46. F. J. Dyson Three identities in combinatorial analysis *J. London Math. Soc.* 18 (1943), 35-39.

47. ——— A new symmetry of partitions J. *combinatory* Theory 7 (1969), 56-61.

48. H. Gollnitz Einfache Partitionen. (Unpublished) Diplomarbeit W. S. 1960, Gottingen.

49. B. Gordon A Combinatorial generalization of Rogers-Ramanujan identities. *Amer. J. Math.* 83 (1961) 393-399.

50. ———Some continued fractions of Rogers-Ramanujan type. *Duke. Math. J.* 31 (1965), 741-748.

51. G. H. Hardy Ramanujan, Cambridge University Press, Cambridge (1940).

52. F. H. Jackson Examples of generalization of Euler's transformation for power series. *Messenger of Math.* 57 (1928), 169-187.

53. V. K. Jain Some transformations of basic hypergeometric series and their applications. *Proc. Amer. Math. Soc.* 78 (1980), 375-384.

54. ——— Some transformations of basic hypergeometric series. *SIAM J. Math. Anal.* 12 (1982).

55. V. K. Jain Certain transformations of basic hypergeometric series and their applications. *Pacific J. Math.* 100 (1982), 1-18.

56. V. Kac Infinite dimensional algebras, Dedekind's η-function classical Mobius functions and the very strange formula. *Advances in Math.* 30 (1978), 85-135.

57. D. H. Lehmer Two non-existence theorems of partitions. *Bull. Amer. Math. Soc.* 52(1946), 538-544.

58. J. Lehner A partition function connected with modulus 5. *Duke Math. J.* 8(1941), 631-635.

59. J. Lepowski Lie algebras and the Rogers-Ramanujan identities. *Notices AMS.* 26(1979), A-457.

60. J. Lepowski and S. Milne Lie algebras and classical partition identities. *Proc. Nat. Acad. Sc. USA* 75(1978), 578-579.

61. I. G. Macdonald Afine root system and Dedekinds η-function. *Inv. Math.* 15(1972), 91-143.

62. L. J. Rogers On the expansion of certain infinite products. *Proc. London Math. Soc.* (1) 24(1893), 337-352.

63. ——— Second memoir on the expansion of certain infinite products. *Proc. London Math. Soc.* (1) 25(1894), 318-343.

64. ——— Third memoir on the expansion of certain infiinite products. *Proc. London. Math. Soc.* (1) 26(1895), 15-32.

65. ——— On two theorems of combinatory analysis and allied identities *Proc. London Math. Soc.* 16(1917), 315-336.

66. L. J. Rogers and S. Ramanujan Proof of certain identities in combinatory analysis. *Proc. Cambridge Phil. Soc.* 19(1919), 211-214.

67. I.J. Schur Ein Beitrag zur additiven Zahlen theorie, Sitzungsber, Akad. *Wissensch*, Berlin, Phys. Math. Klasse (1917), 302-321.

68. Zur additiven Zahlen theorie. sitzungsber. *Akad. Wissensch Berlin Phys. Math. Klasse* (1926), 488-495.

69. A. Selberg Uber einge arithmetische identitaten. Avhandinger Norske Akad. 8(1936).

70. D. B. Sears On the transformation Theory of basic hypergeometric series. *Proc. London Math. Soc.* (2) 53(1951), 158-180.

71. ——— Transformations of basic hypergeometric functions of any order *Proc. London Math. Soc.* (2) 53(1951), 181-191.

72. ——— Transformations of basic hypergeometric series of special type. *Proc. London Math. Soc.* (2) 52 (1951), 467-483.

73. V. N. Singh Certain generalised hypergeometric identities of Rogers-Ramanujan type. *Pacific J. Math.* 7 (1957), 1011-1014.

4

Ramanujan's Contribution to Continued Fractions

Michael D. Hirschhorn

In letters from an "unknown Hindu clerk", G.H. Hardy, at that time (1913) Cayley Lecturer in Mathematics and Fellow of the Royal Society, received the following theorems, among others.

(1) If $u=\dfrac{x}{1+}\dfrac{x^5}{1+}\dfrac{x^{10}}{1+}\dfrac{x^{15}}{1+\ldots}$ and $v=\dfrac{x^{1/5}}{1+}\dfrac{x}{1+}\dfrac{x^2}{1+}\dfrac{x^3}{1+\ldots}$

then $v^5=u\dfrac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}$.

( ) $\dfrac{1}{1+}\dfrac{e^{-2\pi}}{1+}\dfrac{e^{-4\pi}}{1+\ldots}=\{\sqrt{(\dfrac{5+\sqrt{5}}{2})}-\dfrac{\sqrt{5}+1}{2}\}e^{2\pi/5}$.

(3) $\dfrac{1}{1+}\dfrac{e^{-2\pi\sqrt{5}}}{1+}\dfrac{e^{-4\pi\sqrt{5}}}{1+\ldots}=\left\{\dfrac{\sqrt{5}}{1+\sqrt[5]{\{5^{3/4}(\dfrac{\sqrt{5}-1}{2})^{5/2}-1\}}}-\dfrac{\sqrt{5}+1}{2}\right\}e^{2\pi/\sqrt{5}}$.

These, he said[1], "defeated me completely; I had never seen anything like them before. A single look at them is enough to show that they could only be written down by a mathematician of

74. ——— Certain generalised hypergeometric identities of Rogers-Ramanujan type II. *Pacific. J. Math.* 7 (1957), 1691-1699.

75. ——— A note on the computation of Alder's polynomials. *Pacific J. Math.* 9 (1959), 271-275.

76. L. J. Slater A new proof of Rogers transformation of infinite series. *Proc. London. Math. Soc.* (2) 53 (1951), 460-475.

77. ——— Further identities of Rogers-Ramanujan type. *Proc. London Math. Soc.* (2) 54 (1952), 147-167.

78. A. Verma Certain summation formulae for basic hypergeometric series. *Cannd. Math. Bull.* 20 (1977), 369-375.

79. ——— On identities of Rogers-Ramanujan type. *Indian J. Pure Appl. Math.* 11 (1980), 770-790.

80. A. Verma and V. K. Jain Transformations between basic hypergeometric series on different bases and identities of Rogers-Ramanujan type. *J. Math. Anal. Appli.* 76 (1980), 230-269.

81. ——— Transformations of non-terminating basic hypergeometric series, their contour integrals and applications to Rogers-Ramanujan identities. *J. Math. Anal. Appl.* 87 (1982), 9-44.

82. G. N. Watson A new proof of Rogers-Ramanujan identities. *J. London Math. Soc.* 4 (1929), 4-9.

George E. Andrews [14,15] has proved most, if not all, of the algebraic results concerning the Rogers-Ramanujan continued fraction which appear in the "lost" notebook, and has promised to prove the arithmetic results. (Is it too much to hope that in this Centenary year, the funds may be found to publish, and thereby make more readily accessible, all the as yet unpublished manuscripts of Ramanujan, described by R.A. Rankin[16] and K.G. Ramanathan[17] ?

Thus it must seem to the reader that the book has been closed on Ramanujan's work. But on the contrary, a much bigger book lies open before us, with many blank pages. A great opportunity presents itself to us of extending and generalising Ramanujan's results, and of setting them in context. To extend the metaphor of R. Askey, A.J. van der Poorten[18] and others, we have been shown to gems unearthed by Ramanujan, now we must examine the substrate whence they came, and perhaps discover our own gems.

Ramanujan himself hints at this in his second letter to Hardy, when he says concerning result (3) "The above theorem is a particular case of a theorem on the continued fraction

$$\frac{1}{1+}\,\frac{ax}{1+}\,\frac{ax^2}{1+}\,\frac{ax^3}{1+}\,\frac{ax^4}{1+}\,\frac{ax^5}{1+}\,\cdots,$$

which is a particular case of the continued fraction

$$\frac{1}{1+}\,\frac{ax}{1+bx+}\,\frac{ax^2}{1+bx^2+}\,\frac{ax^3}{1+bx^3+}\,\cdots,$$

which is a particular case of a general theorem on continued fractions."

I intend now to spend some time discussing these and related continued fractions. In a short sequence of papers[19,20,21], I was led to consider the cotinued fraction

$$1+a+b+\frac{cq-a}{1+a+bq+}\,\frac{cq^2-a}{1+a+bq^2+}\,\cdots$$

the highest class." Clearly, these theorems played a crucial role in forming Hardy's high opinion of Srinivasa Ramanujan, which led to Ramanujan's coming to England, and subsequently to his great fame.

It is surprising, then, to realise that Ramanujan never published a paper devoted to continued fractions; indeed the only mention of a continued fraction in his published papers of which I am aware is of the Rogers Ramanujan continued fraction (which we refer to later, and of which (1)—(3) are consequences) in 'Proof of certain identities in Combinatory Analysis'[2]. Apart from the marvellous results in his letters to Hardy, we find his principal contributions in the area of continued fractions in Chapters 12 and 16 of his second notebook[3], and in the "lost" notebook."

Although there are far too many theorems to quote here, a few may suffice to convey the scope, the variety, the strangeness, the wonderfulness of his work. Thus, in Chapter 12, which is devoted mainly to continued fractions given by hypergeometric series and gamma-functions, we find the result G.N. Waston[5] describes as "Ramanujan's favourite",

(4) If $a<b$, then

$$\alpha = \frac{ab}{a+b+d-} \frac{(a+d)(b+d)}{a+b+3d-} \frac{(a+2d)(b+2d)}{a+b+5d-\ldots},$$

as well as the result described by Berndt et al.[6] as "certainly one of Ramanujan's crowning achievements in the theory of continued fractions".

(5) If $P=\prod \Gamma\left(\frac{a\pm\beta\pm\gamma\pm\delta\pm\epsilon+1}{2}\right)$ (0, 2 or 4 minus signs)

$$Q=\prod \Gamma\left(\frac{a\pm\beta\pm\gamma\pm\delta\pm\epsilon+1}{2}\right) \text{ (1 or 3 minus signs)},$$

with eight gamma-functions in each product,
then

$$\frac{P-Q}{P+Q}=\frac{8a\beta\gamma\delta\epsilon}{\{2(1+\Sigma a^4)-(1-\Sigma a^2)^2-2^2\}+}$$

$$\cfrac{64\prod(a^2-1^2)}{3\{2(1+\Sigma a^4)-(5-\Sigma a^2)^2-6^2\}+\cfrac{64\prod(a^2-2^2)}{5\{2(1+\Sigma a^4)-(15-\Sigma a^2)^2-14^2\}+\ \ldots.}}.$$

In Chapter 16, which is devoted mainly to what is now known as the theory of q-series, or basic hypergeometric series, we find[7]

$$(6)\quad \frac{(a^2q^3;q^4)_\infty\,(b^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty\,(b^2q;q^4)_\infty}=\frac{1}{1-ab+}\,\frac{(a-bq)\,(b-aq)}{(1-ab)\,(1+q^2)+}\,\cfrac{(a-bq^3)\,(b-aq^3)}{(1-ab)\,(1+q^4)+\ \ldots.}$$

while in the "lost" notebook, Ramanujan gives[8]

$$(7)\quad 1+\frac{aq+\lambda q}{1+}\,\frac{bq+\lambda q^2}{1+}\,\frac{aq^2+\lambda q^3}{1+}\,\frac{bq^2+\lambda q^4}{1+}\ \ldots.=$$

$$=\sum_{n=0}^{\infty}\frac{q^{(n^2+n)/2}\,a^n(-\lambda/a)_n}{(q)_n(-bq)_n}\Big/\sum_{n=0}^{\infty}\frac{q^{(n^2+3n)/2}\,a^n(-\lambda/a)_n}{(q)_n(-bq)_n}.$$

(We shall return to a discussion of (7) later.)

Proofs of the more striking results in the letters, among which I include (1)—(3), as well as the following,

$$(8)\quad \text{If } v=\frac{x}{1+}\,\frac{x^3+x^6}{1+}\,\frac{x^6+x^{12}}{1+}\,\frac{x^9+x^{18}}{1+}\ \ldots$$

$$\text{then } x\left(+\frac{1}{v}\right)=\sum_{n=0}^{\infty}x^{n(n+1)/2}\Big/\sum_{n=0}^{\infty}x^{9n(n+1)/2}$$

$$\text{and } x^3\left(1+\frac{1}{v^3}\right)=\left\{\sum_{n=0}^{\infty}x^{3n(n+1)/2}\Big/\sum_{n=0}^{\infty}x^{9n(n+1)/2}\right\}4$$

were given by Watson[8,9] (who also examines the questions of how Ramanujan may have found these results); simple proofs of (1)—(3) and more in a similar vein given recently by K.G. Ramanathan[10,11].

As part of his monumental work of editing Ramanujan's notebooks, Bruce C. Berndt[12,13], in collaboration with others, has presented and proved, or given a reference to a proof of, everything in Chapters 12 and 16 of the second notebook.

There are many other equally fascinating special cases of (9) and (10), such as[25]

$$(12)\qquad 1+q+\frac{q^2}{1+q^3+}\ \frac{q^4}{1+q^5+\cdots}=\frac{(q^3;q^8)_\infty\ (q^5;q^8)_\infty}{(q;q^8)_\infty\ (q^7;q^8)_\infty}.$$

I believe that (9) or (10) or a close relative could be the "general theorem" that Ramanujan had in mind.

I now give a proof of (9) :

$$\text{Let } c_n=\frac{q^{(n^2-n)/2}b^n\ (-cq/b)_n}{(q)_n\ (aq)_n}.$$

$$\text{Then } (1-q^{n+1})\ (1-aq^{n+1})c_{n+1}=q^n\ (b+cq^{n+1})\,c_n\ ,$$

$$\text{or, } c_{n+1}=q^{n+1}c_{n+1}+aq^{n+1}c_{n+1}-aq^{2n+2}\ c_{n+1}+$$

$$bq^n\ c_n\ +cq^{2n+1}\ c_n\ .$$

put $n=0,1,2,\ldots$ and add, then add c_0 to both sides.

If we let $P(a,b,c,q)=\sum_{n=0}^{\infty} c_n$, then we find

$$P(a,b,c,q)=(1+a+b)\,P(a,bq,cq,q)+(cq-a)P(a,bq^2,cq^2,q)$$

or,

$$\frac{P(a,b,c,q)}{P(a,bq,cq,q)}=1+a+b+\frac{cq-a}{\left(\frac{P(a,bq,cq,q)}{P(a,bq^2,cq^2,q)}\right)}$$

Formula (9) now follows by iteration.

There are also finite versions of (9) and (10); space permits only the finite version of (9). Thus

$$(13)\quad 1+a+b+\frac{cq-a}{1+a+bq+}\ \frac{cq^2-a}{1+a+bq^2+\cdots}\ \frac{cq^N-a}{1+a+bq^N}=\frac{P_N(a,b,c,q)}{P_{N-1}(a,bq,cq,q)},$$

and showed that

$$(9)\quad 1+a+b+\frac{cq-a}{1+a+bq+}\ \frac{cq^2-a}{1+a+bq^2+}\cdots = \sum_{n=0}^{\infty}\frac{q^{(n^2-n)/2}\,b^n(-cq/b)_n}{(q)_n(aq)_n} \Big/ \sum_{n=0}^{\infty}\frac{q^{(n^2+n)/2}\,b^n(-cq/b)_n}{(q)_n(aq)_n}.$$

The numerator and denominator of the right-hand-side can be transformed using Watson's q-analogue of Whipple's theorem[22] to give[23]

$$(10)\quad 1+a+b+\frac{cq-a}{1+a+bq+}\ \frac{cq^2-a}{1+a+bq^2+}\cdots = \frac{1+\sum_{n=1}^{\infty}\frac{(1-cq^{2n})}{(1-cq^n)}\frac{a^n(c/a)_n b^n}{(aq)_n}\frac{(-cq/b)_n}{(-b)_n}\frac{(cq)_n}{(q)_n}q^{(3n^2-n)/2}}{\sum_{n=0}^{\infty}(1-cq^{2n+1})\frac{a^n(cq/a)_n b^n}{(aq)_n}\frac{(-cq/b)_n}{(-b)_{n+1}}\frac{(cq)_n}{(q)_n}q^{(3n^2+n)/2}}.$$

One special case of (10) of particular interest is given by $a=b=0$, $c=1$, and that is

$$(11)\quad 1+\frac{q}{1+}\ \frac{q^2}{1+}\ \frac{q^3}{1+}\cdots = \frac{1+\sum_{n=1}^{\infty}(1+q^n)(-1)^n q^{(5n^2-n)/2}}{\sum_{n=0}^{\infty}(-q^{2n+1})(-1)^n q^{(5n^2+3n)/2}}$$

$$= \frac{1+\sum_{n=1}^{\infty}(-1)^n\left(q^{(5n^2-n)/2}+q^{(5n^2+n)/2}\right)}{1+\sum_{n=1}^{\infty}(-1)^n\left(q^{(5n^2-3n)/2}+q^{(5n^2+3n)/2}\right)}$$

$$= \frac{(q^2;q^5)_\infty\ (q^3;q^5)_\infty\ (q^5;q^5)_\infty}{(q;q^5)_\infty\ (q^4;q^5)_\infty\ (q^5;q^5)_\infty}$$

$$= \frac{(q^2;q^5)_\infty\ (q^3;q^5)_\infty}{(q;q^5)_\infty\ (q^4;q^5)_\infty}$$

which is the spectacularly beautiful and justly celebrated Rogers Ramanujan continuied fraction, referred to earlier.

where[26]

$$P_N(a,b,c,q) = \sum_{r,s,t \geqslant 0} a^r\, b^s\, c^t\, q^{(s^2-s)/2+st+t^2} \begin{bmatrix} r+s+t \\ r \end{bmatrix} \begin{bmatrix} s+t \\ s \end{bmatrix} \begin{bmatrix} N+1-r-t \\ s+t \end{bmatrix}$$

is a polynomial in a, b, c, q. This theorem can easily be proved in the same way as (9).

Now I want to turn my attention to the continued fraction from the "lost" notebook given as (7) above. Once again, the numerator and denominator can be transformed using Watson's theorem mentioned earlier to give[27]

$$(14)\quad 1+\frac{aq+\lambda}{1+}\,\frac{bq+\lambda q^2}{1+}\,\frac{aq^2+\lambda q^3}{1+}\,\frac{bq^2+\lambda q^4}{1+}\,\cdots = \frac{1+\sum_{n=1}^{\infty}\frac{(1-\lambda q^{2n})}{(1-\lambda q^n)}\,\frac{a^n(-\lambda/a)_n}{(-aq)_n}\,\frac{b^n(-\lambda/b)_n}{(-bq)_n}\,\frac{(\lambda q)_n}{(q)_n}(-1)^n q^{(3n^2+n)/2}}{\sum_{n=0}^{\infty}(1-\lambda q^{2n+1})\,\frac{a^n(-\lambda/a)_n}{(-aq)_{n+1}}\,\frac{b^n(-\lambda/b)_n}{(-bq)_n}\,\frac{(\lambda q)_n}{(q)_n}(-1)^n q^{(3n^2+n)/2}}$$

This continued fraction also has many spectacular special cases, of which I mention just one[28],

$$(15)\quad 1+\frac{q+q^2}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^3+q^6}{1+}\,\cdots = \frac{(q^3;q^6)_\infty^2}{(q;q^6)_\infty\,(q^5;q^6)_\infty}.$$

I now give a proof of (7).

Let $$P(a,b,\lambda,q) = (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}\, a^n\, (-\lambda/a)_n}{(q)_n\,(-bq)_n}.$$

It is easy to show using standard theorems from q-series[29] that

$$P(a,b,\lambda,q) = \sum_{r,s,t \geqslant 0} \frac{a^r\, b^s\, \lambda^t\, q^{(r^2+r)/2+(s^2+s)/2+rs+rt+t^2}}{(q)_r\,(q)_s\,(q)_t}.$$

It follows that

$$P(a,b,\lambda,q) = P(b,a,\lambda,q)$$

and also that

$$P(a,b,\lambda,q) = P(aq,b,\lambda q,q) + (aq+\lambda q)P(aq,bq,\lambda q^2,q).$$

Combining these, we have

$$\frac{P(a,b,\lambda,q)}{P(b,aq,\lambda q,q)} = 1 + \frac{aq+\lambda q}{\left(\frac{P(b,aq,\lambda q,q)}{P(aq,bq,\lambda q^2,q)}\right)}.$$

Iteration now gives

$$\frac{P(a,b,\lambda,q)}{P(b,aq,\lambda q,q)} = 1 + \frac{aq+\lambda q}{1+} \; \frac{bq+\lambda q^2}{+1} \; \frac{aq^2+\lambda q^3}{1+} \; \frac{bq^2+\lambda q^4}{1+} \; \ldots,$$

or, on using the symmetry again, and cancelling the factor $(-bq)_\infty$ from numerator and denominator,

$$1 + \frac{aq+\lambda q}{1+} \; \frac{bq+\lambda q^2}{1+} \; \frac{aq^2+\lambda q^3}{1+} \; \frac{bq^2+\lambda q^4}{1+} \; \ldots = \frac{P(a,b,\lambda,q)}{P(aq,b,\lambda q,q)}$$

$$= \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2} a^n (-\lambda/a)_n}{(q)_n (-bq)_n} \Big/ \sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2} a^n (-\lambda/a)_n}{(q)_n (-bq)_n},$$

as required.

once again, there is a finite version of (7), and that is

$$(16) \quad 1 + \frac{aq+\lambda q}{1+} \; \frac{bq+\lambda q^2}{1+} \; \ldots \; \frac{aq^N + \lambda q^{2N-1}}{1} = \frac{P_{2N-1}(a,b,\lambda,q)}{P_{2N-2}(b,aq,\lambda q,q)},$$

$$1 + \frac{aq+\lambda q}{1+} \; \frac{bq+\lambda q^2}{1+} \; \ldots \; \frac{bq^N + \lambda q^{2N}}{1} = \frac{P_{2N}(a,b,\lambda,q)}{P_{2N-1}(b,aq,\lambda q,q)},$$

where

$$P_N(a,b,\lambda,q) = \sum_{r,s,t \geqslant 0} a^r \, b^s \, \lambda^t \, q^{(r^2+r)/2+(s^2+s)/2+rs+rt+st+t^2} \times \begin{bmatrix} N+1-r-s-t \\ t \end{bmatrix} \begin{bmatrix} (N+1)/2]-s-t \\ r \end{bmatrix} \begin{bmatrix} [N/2]-r-t \\ s \end{bmatrix},$$

the sum being taken over all r,s,t with $r+s+t \leqslant [(N+1)/2]$, and where for the present purpose, $\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 1$.

Formula (16) is, perhaps surprisingly, not difficult to prove.

The polynomials $P_n(a,b,\lambda,q)$ that appear in (16) as well as those that appear in (13), satisfy three-term recurrence relations, which lead to the respective finite continued fractions.

Thus we see that the continued fractions given by Ramanujan lead to a study of such polynomials, and perhaps to new and different problems. In this way, the pages of the big and empty book begin to fill....

References

1. G.H. Hardy, The Indian mathematician Ramanujan, Amer. Math. Monthly 44(1937), 137-155, reprinted as Lecture I in G.H. Hardy, Ramanujan, Twelve lectures suggested by his life and work, C.U.P. (1940), reprinted by Chelsea, New York (undated).

   Hardy, on p. 9, may mislead the reader into believing all the formulae he has quoted are from Ramanujan's first letter. However, formula (3) appeared only in Ramanujan's *second* letter to Hardy; see Collected Papers of Srinivasa Ramanujan, edited by G.H. Hardy, P.V. Seshu Aiyar and B.M. Wilson, C.U.P. (1927), reprinted by Chelsea, New York (1962), p. xxviii.
2. Collected Papers, Paper 26 and Appendix I.
3. Notebooks of Srinivasa Ramanujan, Tata Institute of Fundamental Research, Bombay, 1957, Vol. II.

   See also G.N. Watson, Ramanujan's note books, J. London Math. Soc. 6(1931), 137-153 and Bruce C. Berndt, Ramanujan's notebooks, Math. Mag. 51(1978), 147-164.
4. The "lost notebook", so-called by George E. Andrews, is a manuscript (Add. Ms. a. 94[15]) discovered by him in the library of Trinity College, Cambridge. See George E. Andrews, An introduction to Ramanujan's "lost" notebook, Amer. Math. Monthly 86(1979), 89–107.

5. See footnote 3.
6. See footnote 3.
7. A word on notation :

$$(a;q^k)_n = (1-a)(1-aq^k)\ldots(1-aq^{(n-1)k}) \text{ for } n>0,$$

$$(a;q^k)_0 = 1$$

$$(a)_n = (a;q)_n .$$

Thus the left-hand-side of (6) is

$$\prod_{n=0}^{\infty} \frac{(1-a^2q^{4n+3})(1-b^2q^{4n+3})}{(1-a^2q^{4n+1})(1-b^2q^{4n+1})} .$$

8. G.N. Watson, Theorems stated by Ramanujan VII, Theorems on continued fractions, J. London Math. Soc. 4(1929), 39-48.
9. G.N. Watson, Theorems stated by Ramanujan IX, Two theorems on continued fractions, J. London Math. Soc. 4 (1929), 231-237.

   Watson points out that the last part of result (8) is mis-stated in the Collected Papers, p. xxviii.
10. K.G. Ramanathan, On Ramanujan's continued fraction, Acta Arith. XLIII (1984), 209-226.
11. K.G. Ramanathan, on the Rogers Ramanujan continued fraction, Proc. Indian Acad. Sci. Math Sci 93(1984), 67-77.
12. Bruce C. Berndt, Robert L. Lamphere, and B.M. Wilson, Chapter 12 of Ramanujan's second notebook : continued fractions, Rocky Mountain J. Math. 15(1985), 235-310, and R. Askey, Appendix, 311-318.
13. C. Adiga, B.C. Berndt, S. Bhargava and G.N. Watson, Chapter 16 of Ramanujan's second notebook : Theta-functions and q-series. Mem. Amer. Math. Soc. 315(1985).
14. See footnote 4.
15. George E. Andrews, Ramanujan's "lost" notebook III : The Rogers-Ramanujan continued fraction, Adv. in Math. 41(1981), 186-208.

16. R.A. Rankin, Ramanujan's manuscripts and notebooks, Bull. London Math. Soc. 14(1982), 81 97.

17. K.G. Ramanathan, The unpublished manuscripts of Srinivasa Ramanujan, Current Science 50(1981), 203-210.

18. R.A. Askey, MR 85a:01043, MR 86e:33004,
A.J. van der Poorten, MR 85e:11074.

19. M.D. Hirschhorn, Partitions and Ramanujan's continued fraction, Duke Math. J. 39(1972), 789-791.

20. M.D. Hirschhorn, A continued fraction of Carlitz, Duke Math. J. 40 (1973), 77-80.

21. M.D. Hirschhorn, A continued fraction, Duke Math. J. 41 (1974), 27-33.

22. G.N. Watson, A new proof of the Rogers-Ramanujan identities, J. London Math. Soc. 4 (1929), 4-9.

23. This result has not previously been published. It appears as (2.1.3) in my Ph. D. thesis, Developments in the theory of partitions, U.N.S.W., 1979.

24. Here we use the Jacobi triple product identity,

$$(-aq;q^2)_\infty (-a^{-1}q;q^2)_\infty (q^2;q^2)_\infty = \sum_{-\infty}^{\infty} a^n q^{n^2}.$$

25. Basil Gordon, Some continued fractions of the Rogers-Ramanujan type, Duke Math. J. 32(1965), 741-748.

26. Some more notation :

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(q)_n}{(q)_r (q)_{n-r}} \text{ for } 0 \leqslant r \leqslant n,$$

$$= 0 \text{ otherwise.}$$

27. M.D. Hirschhorn, A continued fraction of Ramanujan, J. Austral. Math. Soc. 29(1980), 80-86.

28. See footnote 25.

29. George E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and its Applications, Gian-Carlo Rota, Editor, Addison-Wesley, Reading, Mass., 1976,, esp. Chapter 2.
Dr. Michael Hirschhorn, Department of Mathematics University of New South Wales, Kensungton N.S.W. Australia.

5

Ramanujan's Work on Hypergeometric Series and Integrals

M.A. Pathan

1. Introduction : Ramanujan was a remarkable mathematician in many ways. His work still seems original. As a good indication of this originility we can consider the 'last note book' that Andrews discovered [1] and for which Askey [2] remarked "This work contains hundreds of results Ramanujan found; primarily, and possibly completely, in the last year of his life. Very few of these results have been rediscovered in the more than half a century between when Ramanujan discovered them and when his pages were discovered". There are three famous note books left by Ramanujan [29] other than this "lost notebook". The first and second notebook seem to contain substantially similar material, the second notebook being perhaps Ramanujan's edited and improved version of the first. The third notebook is short and comparatively scrappy. During the past 65 years these notebooks have formed the basis for numerous papers by many mathematicians, including Hardy, Watson, Rogers, Bailey, Andrews and Askey. No course examining the milestones of hypergeometric series would dare omit these giants. The work of Ramanujan on hypergeometric series seem to be of sufficient intrinsic interest to research workers in the theory of series and integrals.

The wide area of summation of series and integrals in which Ramanujan did important research is one of the central parts of the work of Ramanujan. His first paper submitted for publication in Europe [23] in 1914, dealt with this fundamental area of Mathematics, and he returned to this topic again and again, writing atleast six papers ([3] to [8]) on it. All of Ramanujan's manuscripts passed through Hardy's hands. Some of his papers were written in collaboration with Hardy and it was Hardy and Little-wood, J.E. (1885-1977) who built up the great British school of analysis, second to none in the world.

Hardy has the following to say about the work of Ramanujan on hypergeometric series in [13] : "The subject suited him exactly, and here he was unquestionably one of the great masters". Again, Hardy in [16, p. 103] writes : "It was his insight into algebraical formulae, transformations of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal and I can compare him only with Jacobi and Euler." Hardy describes Ramanujan's work in this field in Chapter VII of [13] and [12]. Ramanujan's notebooks were the theme of a lecture [36] given by G.N. Watson to the London Mathematical Society in 1931. It will be noticed that I have left out part of their work, and by those who have already read [12], [13], [16], [36] that I do not duplicate much of their material.

Chapter X and XI of Ramanujan's notebooks are mainly devoted to hypergeometric series, and they are more or less equivalent to Chapter XII of the first edition, of which Professor Hardy has already published a very complete analysis [13]. Watson in [36] remarked that "Chapter X opens with Dougall's theorem in all its generality, springing, as Sylvester might have said, full grown like Athene from the head of Zeus". Chapter XIII of the first edition contains a number of formulae for the linear ttansformations of the ordinary hypergeometric functions ${}_2F_1$ which Ramanujan regarded as corolleries of those of Chapter XII.

If we go through the work of Ramanujan on series then we find that his work deals with some series which resemble his definite integrals not merely in form but in many other respects.

$$ {}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix}\right)=1+\frac{\alpha_1\alpha_2\alpha_3}{1.\beta_1\beta_2}+\frac{\alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1)\alpha_3(\alpha_3+1)}{1.2.\beta_1(\beta_1+1)\beta_2(\beta_2+1)}+\ldots. \quad \ldots.(2) $$

and therefore (1) is a special case of a transformation due to Thomae [23], which was rediscovered by Ramanujan (See Hardy [12]]. The general transformation is

$$ {}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix}\right)=\frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)}\,{}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\beta_2-\alpha_3\\ \alpha_1+\alpha_2-\beta_1+1,\beta_2\end{matrix}\right) $$

$$ +\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+\alpha_2-\beta_1)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)} $$

$$ {}_3F_2\left(\begin{matrix}\beta_1-\alpha_1,\beta_2-\alpha_2,\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3\\ \beta_1-\alpha_1-\alpha_2+1,\beta_1+\beta_2-\alpha_1-\alpha_2\end{matrix}\right) \quad \ldots.(3) $$

which is valid when R $(\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2)<0$, $R(\alpha_3-\beta_1+1)>0$

In (3), it is essential to make $\alpha_3=n$ before making $\beta_1\to n+1$ in order to reduce Thomae's transformation to a relation between two series only.

In a paper [39], Whipple examined the formula (1) of Ramanujan which is the kernel of Hardy's paper [12] and he showed that Ramanujan's formula can be derived from two fundamental relations of Thomae [33] published in 1879.

$$ {}_3F_2\left(\begin{matrix}a,b,c\\ c,f\end{matrix}\right)=\frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\,{}_3F_2\left(\begin{matrix}e-a,\ f-a,\ s\\ s+b,\ \ s+c\end{matrix}\right) $$

when s is equivalent to $e+f-a-b-c$, and

$$ {}_3F_2\left(\begin{matrix}a,b,c\\ e,f\end{matrix}\right)=\frac{\Gamma(1-a)\Gamma(e)\Gamma(f)\Gamma(c-b)}{\Gamma(e-b)\Gamma(f-b)\Gamma(1+b-a)\Gamma(c)}\,{}_3F_2\left(\begin{matrix}b,b-e+1,b-f+1\\ 1+b-c,1+b-a\end{matrix}\right) $$

+a similar expression obtaind by interchanging b and c.

Hardy [12] has given a catalogue of all the known cases in which a series ${}_3F_2$ with last element unity is expressible in terms of Gamma function. Incidentally, he quoted a formula which Watson discovered, namely that

In the case of integrals there are special values of a parameter for which the integrals may be eveluated infinite terms. For example, his paper 'On certain infinite series' [25] is merely a continuation of the paper on 'Some definite integral' [23]. In this context, it is notable that in certain series the corresponding results involve Elliptic functions. Given this rather general and simple basis, it is pleasant surprise to see the extent to which a variety of full or partial generalisations of results of Ramanujan's integrals may be proved. A very important role is played by Askey's work [2] on Ramanujan's integrals [23] which are extensions of the beta functions given as an integral on $[0,\infty)$ and the sum again contains a few of Jacobi's elliptic function identities.

The scope of this article is modest : we will almost exclusively be concerned with Ramanujan's work on hypergeometric series and definite integrals and their relation to that of earlier and later research workers. The author's primary concern has been to weave Ramanujan's published and unpublished papers into a coherent theory of hypergeometric functions. Basic hypergeometric series (or q series) and q-integrals will not be considered.

2. **Series** : The last topic with which Ramanujan dealt in his letter of 16 January, 1913, to Hardy, was that of divergent series. We quote from page 351 of the *collected papers* [30] : "I have also given meanings to the fractional and negative number of terms in a series, and can calculate such values exactly and approximately. Many remarkable results have been got from such theorems". e.g. :

$$\frac{1}{n}+\left(\frac{1}{2}\right)^2\frac{1}{n+1}+\left(\frac{1.3}{2.4}\right)\frac{1}{n+2}+\ldots-\left[\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}\right]^2\cdot$$

$$\left\{1+\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2+\ldots\text{to n terms}\right\}\quad\ldots(1)$$

The left hand side of (1) is equal to

$$\frac{1}{n}\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},n\\1,\ n+1\end{matrix}\right),$$

where the hypergeometric series ${}_3F_2$ is given by

$$ {}_3F_2\left(\begin{matrix}-n,\lambda,2\lambda+2\mu+n-1\\ 2\lambda,\lambda+\mu\end{matrix}\right) = $$

$$ \frac{n\,!\;\Gamma(\lambda+\frac{1}{2}n)\,\Gamma(\mu+\frac{1}{2}n)\,\Gamma(\frac{1}{2}\lambda)\,\Gamma(\lambda+\mu)}{(\frac{n}{2})\,!\;\Gamma(\lambda+\mu+\frac{1}{2}n)\Gamma(2\lambda+n)\,\Gamma(\lambda)\,\Gamma(\mu)} $$

....(4)

where n is an even integer (when n is an odd integer the sum of hypergeometric series is zero). Hardy pointed out that it was a special case of a formula discovered by Ramanujan. Watson has given his own proof of this formula in [35] using a connection between the hypergeometric function and Jacobi polynomials.

Watson [37] raises the question whether Ramanujan's formula (1) can be established by induction. His own proof is by transformation of series, and it seems possible that Ramanujan obtained the theorem in a similar manner. This formula of Ramanujan proved by Watson was the subject of a note by H.B.C. Darling in [8] where he gave two proofs by induction. The theorem is a limiting case of the proposition of Whipple [41] that the sum of n terms of ${}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\right)$ is equal to

$$ \frac{\Gamma(1+\alpha-\gamma)\,\Gamma(1+\beta-\gamma)}{\Gamma(1-\gamma)\,\Gamma(1-\gamma+\alpha+\beta)}\left\{1-\frac{(\alpha)_n(\beta)_n}{n\,!\,(\gamma-1)_n}\,{}_3F_2\left(\begin{matrix}1-\alpha,1-\beta,n\\ 2-\gamma,n+1\end{matrix}\right)\right\}. $$

This result is a generalization of (1) proved in different ways by Watson [37] and Darling [8].

Generalizations of (1) have also been obtained by J. Hodgkinson [18] and W. N. Bailey [5]. Watson has made use of the theorem on the constants of Landau and Labesgue.

Bailey proved a more general result in [6]

$$ \frac{\Gamma(x+m)\;\Gamma(y+m)}{\Gamma(m)\;\Gamma(x+y+m)}\;{}_3F_2\left(\begin{matrix}x,y,v+m-1\\ v,x+y+m\end{matrix}\right) \text{ to n terms} $$

$$ =\frac{\Gamma(x+n)\;\Gamma(y+n)}{\Gamma(n)\;\Gamma(x+y+n)}\;{}_3F_2\left(\begin{matrix}x,y,v+n-1\\ v,x+y+n\end{matrix}\right) \text{ to m terms} \qquad ....(5) $$

When $x=y=\frac{1}{2}$, $v=1$, (5) reduces to the theorem that

$$\left(\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)}\right)^2 \left\{\frac{1}{m}+\left(\frac{1}{2}\right)^2 \frac{1}{m+1}+\left(\frac{1.3}{2.4}\right)^2 \frac{1}{m+2} \text{ ....to n terms}\right\}$$

$$= \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}\Big)^2 \left\{\frac{1}{n}+\left(\frac{1}{2}\right)^2 \frac{1}{n+1}+\left(\frac{1.3}{2.4}\right)^2 \frac{1}{n+2}+\text{.... to m terms}\right\} \quad \text{....(6)}$$

which gives Ramanujan's theorem when m tends to ∞

If we let m tend to ∞ in (5), we find that the sum of n terms of the series

$${}_2F_1\left(\begin{matrix}x,y\\v\end{matrix}\right) \text{ is } \frac{\Gamma(x+n)\ \Gamma(y+n)}{\Gamma(n)\ \Gamma(x+y+n)}\ {}_3F_2\left(\begin{matrix}x,y,v+n-1\\v,x+y+n\end{matrix}\right)$$

which is the first result of Bailey [5] referred to above.

In a paper [6], Bailey gave three proofs of (5). The first has the widest appeal since it assumes no knowledge of transformations of generalized hypergeometric series and is on the lines of Darling's proof of (1), the second is of a different type from any of the earlier proofs of (1) and is probably as simple a proof as one can expect; and the third is the method by which the theorem was discovered. The formula (5) is also an immediate consequence of the result of Bailey [5]

$${}_3F_2\left(\begin{matrix}x,y,z\\v,w\end{matrix}\right) \text{ to n terms } = \frac{\Gamma(v+w+n-1)\,\Gamma(x+n)\,\Gamma(y+n)\,\Gamma(z+n)}{\Gamma(n)\,\Gamma(y+z+n)\,\Gamma(z+x+n)\,\Gamma(x+y+n)}\cdot$$

$${}_7F_6\left(\begin{matrix}a,1+\frac{1}{2}a,\ w+n-1,\ v+n-1,\ x,y,z\\ \frac{1}{2}a,v,w,\ y+z+n,\ z+x+n,\ x+y+n\end{matrix}\right) \quad \text{....(7)}$$

provided that $v+w=x+y+z+1$ and $a = x+y+z+n-1$.

In [38], G.N. Watson discussed five of the problems, numbered (3), (5), (9), (10), (11) in Ramanujan's second letter to Hardy, quoted on pages xxviii and 352 of the *collected papers* [30]. One of the problem is an interesting summation

$$1-5\left(\frac{1}{2}\right)^5+9\left(\frac{1.3}{2.4}\right)^5-13\left(\frac{1.3.5}{2.4.6}\right)^5+\ldots=\frac{2}{[\Gamma(3/4)]^2} \quad \text{....(8)}$$

The formula is apparently not a special case of Dougall's theorem; it can be derived from Whipple's formula [40].

$$ {}_6F_5\left(\begin{matrix} a, 1+\tfrac{1}{2}a, c, d, e, f \\ \tfrac{1}{2}a, 1+a-c, 1+a-d, 1+a-e, 1+a-f \end{matrix}; -1\right) $$

$$ = \frac{\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a)\,\Gamma(1+a-e-f)}\,{}_3F_2\left(\begin{matrix} 1+a-c-d, e, f \\ 1+a-c, 1+a-d \end{matrix}\right) \qquad \ldots(9) $$

by taking $a = c = d = e = f = \frac{1}{2}$. We thus find that the given series is equal to

$$ \frac{2}{\pi}\,{}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}\right) $$

Now using Dixon's formula [9]

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix}\right) = \frac{\Gamma(1+\frac{1}{2}a)\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\,\Gamma(1+\frac{1}{2}a-b)\,\Gamma(1+\frac{1}{2}a-c)\,\Gamma(1+a-b-c)}. \qquad \ldots(10) $$

we have

$$ \frac{2}{\pi}\,{}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}\right) = \frac{2}{[\Gamma(3/4)]^4} $$

whence the formula is obvious.

Another interesting summation of Ramanujan is

$$ 1 - \left(\frac{1}{2}\right)^3 + \left(\frac{1.3}{2.4}\right)^3 - \ldots = \left\{\frac{\Gamma(9/8)}{\Gamma(5/4)\,\Gamma(7/8)}\right\}^2. \qquad \ldots(11) $$

Hardy has shown in [13] that if in the two formulae

$$ \left\{{}_2F_1\left(\begin{matrix} \alpha, \beta \\ \alpha+\beta+\frac{1}{2} \end{matrix}; x\right)\right\}^2 = {}_3F_2\left(\begin{matrix} 2\alpha, \alpha+\beta, 2\beta \\ \alpha+\beta+\frac{1}{2}, 2\alpha+2\beta \end{matrix}; x\right) $$

and

$$ {}_2F_1\left(\begin{matrix} \alpha, \beta \\ 1+\alpha-\beta \end{matrix}; -1\right) = \frac{\Gamma(1+\alpha-\beta)\,\Gamma(1+\frac{1}{2}\alpha)}{\Gamma(1+\alpha)\,\Gamma(1+\frac{1}{2}\alpha-\beta)} $$

due to Clausen and Kummer respectively, we put $\alpha = \beta = \frac{1}{4}$ and $x = -1$, we get (11). Whipple [40] gave two generalizations of (11) in terms of the series.

$${}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}+x, \frac{1}{2}-x \\ 1+x, 1-x\end{matrix}; -1\right) \text{ and } {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}+x, \frac{1}{2}+2x \\ 1+x, 1+2x\end{matrix}; -1\right)$$

which reduces to Ramanujan's series when $x=0$.

H.F. Sandham [31] has shown that Clausen's [7] and Orr's [19] theorems

$$[{}_2F_1\left(\begin{matrix}\alpha, \beta \\ \alpha+\beta+\frac{1}{2}\end{matrix}; x\right)]^2 = {}_3F_2\left(\begin{matrix}2\alpha, 2\beta, \alpha+\beta \\ \alpha+\beta+\frac{1}{2}, 2\alpha+2\beta\end{matrix}; x\right) \quad \ldots(12)$$

$${}_2F_1\left(\begin{matrix}\alpha, \beta \\ \alpha+\beta+\frac{1}{2}\end{matrix}; x\right) {}_2F_1\left(\begin{matrix}\frac{1}{2}-\alpha, \frac{1}{2}-\beta \\ 3/2-\alpha-\beta\end{matrix}; x\right)$$

$$= {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}+\alpha-\beta, \frac{1}{2}-\alpha+\beta \\ \alpha+\beta+\frac{1}{2}, \frac{3}{2}-\alpha-\beta\end{matrix}; x\right) \quad \ldots(13)$$

although 70 years separates their discovery, can be deduced directly form one another on our recalling the classical theorem which for hypergeometric functions of order 3 is that

$${}_2F_1\left(\begin{matrix}a, b \\ a\end{matrix}; x\right), \quad x^{1-d}\, {}_3F_2\left(\begin{matrix}a+1-d, b+1-d, c+1-d \\ 2-d, c+1-d,\end{matrix}; x\right),$$

$$x^{1-e}\, {}_3F_2\left(\begin{matrix}a+1-e, b+1-e, c+1-e \\ d+1-e, 2-e\end{matrix}; x\right)$$

satisfy the same differential equation of the third order with polynomial coefficients.

Also, Sandham [31] deduced Orr's theorem from the Dougall-Ramanujan formula [10].

$${}_7F_6\left(\begin{matrix}a, 1+\frac{1}{2}a, b, c, d, e, -m \\ \frac{1}{2}a, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+m\end{matrix}; 1\right)$$

$$= \frac{(1+a)_m\,(1+a-b-c)_m\,(1+a-b-d)_m\,(1+a-c-d)_m}{(1+a-b)_m\,(1+a-c)_m\,(1+a-d)_m\,(1+a-b-c-d)_m}$$

$\ldots(14)$

where $1+2a=b+c+d+e-m$ and thus these two most interesting theorems on products of hypergeometric functions can be deduced from well-known results, without much labour.

On expanding the hypergeometric function in powers of x and equating coefficients of x^n we see that Orr's theorem is equalent to

$$ {}_4F_3\left(\begin{matrix}\alpha+\beta-n-\frac{1}{2}, \alpha, \beta-n \\ \beta-n+\frac{1}{2}, \alpha-n+\frac{1}{2}, \alpha+\beta+\frac{1}{2}\end{matrix}; 1\right)= $$

$$ =\frac{(\frac{1}{2})_n\,(\frac{1}{2}+\alpha-\beta)_n\,(\frac{1}{2}-\alpha-\beta)_n}{(\frac{1}{2}-\alpha)_n\,(\frac{1}{2}-\beta)_n\,(\alpha+\beta+\frac{1}{2})_n} \qquad \ldots(15) $$

which is a particular case of Dougall-Ramanujan formula.

Among the mathematical notes left by Ramanujan, occurs without proof, the theorem

$$ {}_1F_1(\alpha,\beta;x)\;{}_1F_1(\alpha,\beta;-x)={}_2F_3\left(\begin{matrix}\alpha, \beta-\alpha \\ \beta, \frac{1}{2}\beta, \frac{1}{2}(\beta+1)\end{matrix}; \frac{1}{4}x^2\right) \qquad \ldots(16) $$

C.T. Preece [22(a)] proved this theorem and investigated many more similar theorems on product of two generalized hypergeometric series in terms of ${}_1F_2$, ex ${}_1F_2$, ${}_2F_3$, ex ${}_2F_3$ by using differential equation of order four (in special cases of order three).

In the course of last two decades the formula (16) has received a certain theoretical foundation and has found important applications in the theory of special functions. Many generalizations of (16) have appeared in the literature. Shanker and Saran [32] in 1970, obtained its generalization in which the Kampe de Feriet's series can be expressed in terms of a generalized hypergeometric series. Ideas similar to this have been employed by many classical mathematicians to obtain results of hypergeometric series of 2 and more variables. Using Dixon's theorem. Pathan, Qureshi and Khan [21] obtained a reduction formula of Kampe de Feriet's series into a combination of two generalized hypergeometric series pFq's and deduced some reduction formulae of Bailey [3] and Preece [22(a)] as special cases. In [20], Ramanujan's formula (16) is generalized to give transformations and reduction formulae of generalized Horn function.

There have been published several proofs of Dixon's theorem [9]

$$ {}_3F_2\left(\begin{matrix}a,b,c \\ 1+a-b,\ 1+a-c\end{matrix}\right)= $$

$$ \frac{\Gamma(1+\frac{1}{2}a)\;\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\,\Gamma(1+\frac{1}{2}a-b)\,\Gamma(1+\frac{1}{2}a-c)\,\Gamma(1+a-b-c)}. \qquad \ldots(17) $$

The original proof given by Dixon [9] involved the use of integrals of Pochhammer's type and was long and involved. The

second proof was given by Dougall [10] and independently by Ramanujan [12] both of whom deduced the theorem by a limiting process from Dougall's theorem for the sum of a well-poised terminating series of the type ${}_7F_6$. A simpler and more direct proof was given by Watson [34] who used double series together with the theorem of Gauss and Kummer for summing the ordinary hypergeometric series with arguments ± 1. All these proofs are quite different from one of the standard methods [42] of deducing Gauss theorem from a relation between contiguous hypergeometric series of the ordinary type. Bailey in [4] gave an analogous method of proving Divon's theorem.

The problem of computing finite and infinite sums arising in diverse areas of discrete and continuous mathematics is of special interest in the frame-work of combinational analysis for solving enumerational problems. Besides Ramanujan, a number of remarkable results are due to Pascal, Leibniz, Euler, Gauss, Jacobi, MacMahon, Hardy, Polya, Gould, Rota, Stanley and other workers. (We did not consider such interesting sums due to limited space.) The idea of making systematic use of contour integrals in estimating combinational sums was apparently first worked out by Ramanujan, Hardy and Littlewood in the 'circular wheel' (See [30] or circle method).

3. **Integrals** : The fundamental formula of Chapter VII of [13] is

$$\sum_{n=0}^{\infty} (-1)^n (s+2n) \frac{s^{(n)} (x+y+z+u+2s+1)^{(n)}}{n!\,(x+y+z+u+s)_{(n)}} \prod_{x,y,z,u} \frac{x_{(n)}}{(x+s+1)^{(n)}}$$

$$= \frac{s}{\Gamma(s+1)\,\Gamma(x+y+z+u+s+1)} \prod_{x,y,z,u} \frac{\Gamma(x+s+1)\,\Gamma(y+z+u+s+1}{\Gamma(z+u+s+1)} \qquad \ldots(18)$$

where one of the five arguments x,y,z,u, -x-y-z-u-2s-1 is a positive integer, $a^{(n)} = \Gamma(a+n)/\Gamma(a)$ and $a_{(n)} = \Gamma(a+1)/\Gamma a-n+1)$

Ramanujan seems to have found this formula about 1910 or 1911, but he had been anticipated by Dougall. While looking at the special cases of (18), Hardy [12, p. 496] found that it has an nteresting connection with the theory of Bessel functions. He started from the formula

$$J_a(u)=(\tfrac{1}{2}u)^{a-s}\sum_{n=0}^{\infty}\frac{(s+2n)}{n!}\,\frac{\Gamma(s+n)\,\Gamma(a-s+1)}{\Gamma(a+n+1)\,\Gamma(a-s-n+1)}\,J_{s+2n}(u) \qquad \ldots(19)$$

and obtained (18) by using an integral

$$\int_0^{\infty} J_{\alpha}(u)J_{\beta}(u)\,u^{\lambda}\,du.$$

The application of integrals the whole computation process, and the knowledge of integral representations of special functions makes it possible to compute sums. Using this fact, Prof. Watson communicated a very simple proof of Ramanujan's formula (3) to Hardy [See [12], p. 498]. Watson's proof was based on the integral representation

$${}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix}\right)=\frac{\Gamma(\beta_2)}{\Gamma(\alpha_3)\,\Gamma(\beta_2-\alpha_3)}\int_0^1 x^{\alpha_3-1}\,(1-x)^{\beta_2-\alpha_3-1}\,{}_2F_1\left(\begin{matrix}\alpha_1,\alpha_2\\ \beta_1\end{matrix};x\right)dx. \qquad \ldots(20)$$

Ramanujan's work on series may be considered as continuation of his work on definite integrals [see [25] and [23]]. During 1913, Ramanujan held a research studentship in the University of Madras and was occupied mainly with definite integrals. The theorems stated by Ramanujan in the letters which he wrote to Hardy on 16 January and 27 February, 1913, before leaving India, which have now appeared in full in his *collected papers*, seem to be of sufficient intrinsic interest to research workers. A theorem described as "IV. Theorem on integrals" in the first letter published on page XXV (and 349) of Ramanujan's *collected papers* [see [13], p. 8 eqn (1.5)] is

$$\int_0^{\infty}\frac{1+[x/(b+1)]^2}{1+[x/a]^2}\cdot\frac{1+[x/(b+2)]^2}{1+[x/(a+1)]^2}\ldots dx$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma(a+\frac{1}{2})\Gamma(b+1)\Gamma(b-a+\frac{1}{2})}{\Gamma(a)\Gamma(b+\frac{1}{2})\Gamma(b-a+1)}. \quad \ldots(21)$$

A proof of (21) and a similar result, was published later by Ramanujan [23]. Another result

$$\int_0^\infty \frac{dx}{[1+(x/a)^2][1+\{x/(a+1)\}^2]\ldots[1+(x/b)^2][1+\{x/(b+1)\}^2]}$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})\Gamma(a+b)}{\Gamma(a)\Gamma(b)\Gamma(a+b+\frac{1}{2})} \quad \ldots. (22)$$

was proved in the same paper [23] and has also been proved by Hardy [15]. The method employed was based upon the change of integration in a double integral. C. T. Preece [22(b)] gave a proof of the result which involves only the resolution of an expression into partial fractions and the well known relation between the hypergeometric series and the gamma function. Thus integral is shown equivalent to

$$\frac{\pi}{2} \frac{\Gamma(a+b)}{[\Gamma(a)\Gamma(b)]^2}[\Gamma(2a)\Gamma(b-a){}_2F_1\left(\begin{matrix}2a, a+b\\ a-b+1\end{matrix}\right)\Gamma(2b)\Gamma(a-b){}_2F_1$$

$$\left(\begin{matrix}2a, a+b\\ b-a+1\end{matrix}\right)]$$

which reduces to the expression given in right hand side of (22).

Ramanujan's form of inversion theorem of Chapter XVI [29]. [36, p. 148] is interesting. Using the notations of hypergeometric functions, Watson [36] rewrites this in the form : If

$$F(x) = \exp\left[-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; x)}\right]$$

then

$$F\left(1-\frac{\phi^4(-x)}{\phi^4(x)}\right) = x$$

and

$$\phi^2[F(x)] = 1+(\tfrac{1}{2})^2 x+\left(\frac{1.3}{2.4}\right)^2 x^2+\left(\frac{1.3.5}{2.4.6}\right)^2 x^3+\ldots.$$

where $\phi(x) = 1+2x+2x^4+2x^9+\ldots.$

It is not very difficult to recognise that, if we define the quarter periods K and K′ as definite integrals in the usual manner, and then define q as exp $(-\pi K'/K)$, then these results assert the truth of the formulae K and K′ in terms of Thetafunctions of q.

In Chapter XXI of Ramanujan's notebooks [29], there are developments of functions analogous to elliptic functions which are not seen elsewhere. For example, there is a whole theory built on the basis of the equation.

$$y=\exp\left[-\frac{2\pi}{\sqrt{3}}\,\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3},1;1-x\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3},1;x\right)}\right]$$

in place of the customary equation

$$y=\exp\left[-\pi\,\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;1-x\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;x\right)}\right]$$

and several other similar theories. Of course these theories are deducible from the theory of elliptic functions, but that is not the quickest way of obtaining Ramanujan's results.

The formula given in Chapter IV of notebooks [29]

$$\int_0^\infty x^{n-1}\{\phi(0)-x\phi(1)+x^2\phi(2)-\ldots\}dx=\frac{\pi\,\phi(-n)}{\sin n\pi} \qquad \ldots. (23)$$

is a version of Mellin's formula and is valid only when various limitation are placed on the function $\phi(n)$. Formula (23) leads to various similar formulae, of which Watson [36] quotes

$$\int_0^\infty\left\{\phi(0)-\frac{x}{1!}\phi(1)+\frac{x^2}{2!}\phi(2)-\ldots.\right\}\cos nx\,dx$$

$$=\phi(-1)-n^2\phi(-3)+\ldots. \qquad \ldots.(24)$$

and thence to the result that :

If $\int_0^\infty \phi(x)\cos nx\,dx=\psi(n)$ then $\int_0^\infty \psi(x)\cos nx\,dx=\frac{\pi}{2}\phi(n)$

and

$$\int_0^\infty \psi^2(x)\,dx=\frac{\pi}{2}\int_0^\infty \phi^2(x)\,dx. \qquad \ldots. (25)$$

(25) indicates Ramanujan's approach to Fourier's integrals theorem.

unless n is an integar or a is of the from r^p, where p is a positive positive integer.

Ramanujan stated (26) in [23] on page 16 and started the section containing it with "another curious formula is the following". Hardy gave the first proof of (26) in [15]. For formula (26), Ramanujan writes in [23] : "My own proof of the above result makes use of a general formula, the truth of which depends on conditions which I have not yet investigated completely. A direct proof depending on Cauchy's Theorem will be found in Mr. Hardy's note which follows this paper". He closed this paper with the evaluation of "another curious integral", which is another important integral.

If $a=r^p$, the formula (26) reduces to

$$\int_0^\infty \frac{x^{n-1}\,dx}{(1+x)(1+rx)\ldots(1+r^px)} = \frac{\pi}{\sin n\pi} \frac{(1-r^{1-n})(1-r^{2-n})\ldots(1-r^{p-n})}{(1-r)(1-r^2)\ldots(1-r^p)} \quad \ldots (27)$$

Askey in 1980 remarked in [2] that the identities of Ramanujan of the type (26) are extensions of the beta function given as an integral on $[0, \infty]$ and the sum contains a few of Jacobi's elliptic function identities. He observed that if $x+y=j$ then (27) is a natural extension of the beta integral.

$$\int_0^\infty \frac{t^{x-1}dt}{(1+t)^{x+y}} = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \beta(x, y) \quad \ldots (28)$$

for $j=1,2,\ldots$ and $0<x<j$ and thus (28) is the integral which Ramanujan extended. Ramanujan also found a sum [see (2), p. 351] that contains both the q-binomial theorem and the triple product for the theta functions as special cases. Interest in the Ramanujan's extension of the gamma and beta functions has increased appreciatly in recent times. Clearly some important mathematics is contained in these identities and they are more than "curious".

Ramanujan had learnt Fourier integral theorem from a book of Carr and he had found many of the formal developments of this theory during his studentship at University of Madras. To quote Hardy [17] "He had most of his formal ideas which underlie the recent work of Watson, and of Titchmarsh and myself on Fourier Kernels and reciprocal functions". He could of course evaluate any evaluable definite integral. Thus he knew that, if

$$G(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(t) \cos xt \, dt$$

is the Fourier cosine transform of F(x), and

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty F(x)\, x^{s-1}\, dx; \quad g(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty G(x)\, x^{s-1}\, dx$$

are the 'Mellin transforms' of F(x) and G(x) then

$$g(s) = \sqrt{\frac{2}{\pi}}\, \Gamma(s) \cos \frac{s\pi}{2} f(1-s)$$

or $$\frac{g(s)}{2^{\frac{1}{2}}\, \Gamma(\frac{1}{2} s)} = \frac{f(1-s)}{2^{\frac{1}{2}}\, (1-s)\, \Gamma\frac{1}{2}\, (1-s)}.$$

Conditions for the validity of Ramanujan's formulas and analogouls of formula (24) in which cos x or sin x are replaced by general 'Fourier Kernel' K(x) was examined by Hardy in [14] and a suggestion was made that they might be extended to the more general Watson transforms, with a formal indication of the method to be used. An extension to general Watson transforms and a discussion of a special case closely related to a Helbert transform was later given by F.M. Goodspeed [11].

The following formula of Ramanujan [23] is of considerable interest :

If $o<r<1$, $n>o$ and $o<a<r^{n-1}$, then

$$\int_0^\infty \frac{(1+arx)\,(1+ar^2x)\ldots}{(1+x)\,(1+rx)\,(1+r^2x)\ldots\ldots} x^{n-1}\, dx$$

$$= \frac{\pi}{\sin n\pi} \prod_{m=0}^{\infty} \frac{(1-r^{m-n})\,(1-ar^m)}{(1-r^m)\,(1-ar^{m-n})} \qquad \ldots (26)$$

References

1. Andrews, G.E., An Introduction to Ramanujan's "Lost" note-book, *Amer. Math. Monthly, 86* (1979), 89-108.
2. Askey, R., Ramanujan's extension of the gamma and beta functions, *Amer. Math. Monthly, 87* (1980), 346-359.
3. Bailey, W. N., On a sum of a terminating ${}_3F_2$ (1), *Quart. J. Math. Oxford Ser* (2) *4* (1953), 237-240.
4. ———A new proof of Dixon's theorem on hypergeometric series, *Quart. J. Math. Oxford 8* (1937), 113-114.
5. ———The partial sum of the coefficients of the hyper-geometric series, *J. London Math. Soc. 6* (1931), 40 41.
6. ———On one of Ramanujan's theorem, *J. London Math. Soc. 7* (1932), 34-36.
7. Clausen, T., Ueber die falle U.S.W., *J. für Math. 3* (1828), 82-95.
8. Darling, H.B.C., On a proof of one of Ramanujan's theorems, *. London Math. Soc. 5* (1930), 8-9.
9. Dixon, A.C., Summation of a certain series, *Proc. London Math. Soc.* (1) 35 (1902), 284-289.
10. Dougall, T., On Vandermondes theorem and some more general expansions, *Proc. Edinburg Math. Soc. 25* (1907), 114-132
11. Goodspeed, F.M., Some generalizations of a formula of Ramanujan, *Quart. J. Math. Oxford, 10* (1939), 210-218.
12. Hardy, G.H., A chapter from Ramanujan's note-book, *Proc. Cambridge Phil. Soc. 21* (1923), 492-503.
13. ———Ramanujan, Cambridge Univ. Press, 1940, Reprinted by Chelsea, New York, 1959.
14. ———Ramanujan and the theory of Fourier transforms, *Quart. J. Math. Oxford 8* (1937), 245-254.
15. ———Proof of a formula of Mr. Ramanujan, *Messenger of Math. 44* (1915), 18-21.

16. ———Srinivasa Ramanujan (Obituary notes), *J. Indian Math. Soc 14* No. 3 (1922), 82-104.

17. ————The Indian Mathematician Ramanujan, *Amer. Math. Monthly, 44* (1937), 137-155.

18. Hodgkinson, J., Note on one of Ramanujan's theorem, *J. London Math. Soc. 6* (1931), 42-43.

19. Orr, M. Mc F., Theorems relating to the product of two hypergeometric series, *Trans, Cambridge Phil. Soc. 17* (1899), 1-15

20. Pathan, M.A. and Qureshi, M.I., on some transformations and reduction formulae of generalized Horn function $H_3(P)$ I, *Soochow J. Math. 8* (1982), 163-170.

21. Pathan, M.A., Qureshi, M.I. and Khan, F.U., Reducibility of Kampe de Feriet's hypergeometric series of higher order, *Serdica, 11* (1985), 20-24.

22 (a). Preece, C.T., The product of two generalized hypergeometric functions, *Proc. London Math.* Soc. (2) *22* (1924) 370-380.

22 (b).————Theorems stated by Ramanujan (1) : Theorems on integrals, *J. London Math. Soc. 3* (1928), 212-216.

23. Ramanujan, S. Some definite integrals, *Messenger of Math. 44* (1915), 10-18.

24. ———. Some definite integrals connected with Gauss sums, *Messenger of Math., 44* (1915), 75-85.

25. ————, On certain infinite series, *Messenger of Math., 45* (1916), 11-15.

26. Ramanujan, S., Summation of certain series, *Messenger of Math.* (1916), 157-160.

27. ————, A class of definite integrals, *Quart J. Math.* Oxford *48* (1920), 294-309.

28. ————, Some definite integrals, *J. Indian Math.* Soc. *11* (1919), 81-88.

29. ————, Note-books of Srinivasa Ramanujan, 2 Vols. T. I. F. R., Bombay, 1957.

30. ———, Collected papers, Combridge Univ. Press London, 1927. Reprinted Chelsea, New York, 1962.
31. Sandham, H.F., A square and a product of hypergeometric functions, *Quart. J. Math.* Oxford (2) I (1956), 153-154.
32. Shanker, O. and Saran S., Reducibility of Kampe de Feriet's function, *Ganita*, *21* no. 1 (1970), 9-16.
33. Thomae, J., Ueber die Funktionen U.S.W., *J. fur Math.*, *87* (1879), 26-73.
34. Watson, G.N.. Dixon's theorem on generalized hypergeometric functions, *Proc. London Math. Soc.* (2) *22* (1924), 32-35.
35. ———, A note on generalized hypergeometric series, *Proc. London Math. Soc.* (2) *23* (1925), 13-15.
36. ———, Ramanujan's note books, *J. London Math.* Soc. *6* (1931), 137-153.
37. ———, Theorems stated by Ramanujan (VIII) : theorems on divergent series, *J. London Math.* Soc. *4* (1929), 82-86.
38. ———, Theorems stated by Ramanujan (XI), *J.London Math. Soc. 6* (1931), 59-65.
39. Whipple, F.J.W., A group of generalized hypergeometric series *Proc. London Math.* (2) *23* (1925), 104-114.
40. ———, On well poised series, generalized hypergeometric series having parameters in pairs, each pair with the same sum, *Proc. London Math.* Soc. (2) *24* (1926), 247-63.
41. ———, The sum of the coefficients of hypergeometric series, *J. London Math.* Soc. *5* (1930), 192.
42. Whittaker, E.T.and Watson, G.N., A course of modern analysis., 4th edition, Cambridge Univ. Press, 1927.

6

Ramanujan and Theta Functions

by

Professor R.P. Agarwal

1. Introduction

In this article, we have briefly described some of the recent work done on a topic in which according to G.H. Hardy 'both the profundity and the limitations of Ramanujan's knowledge stand out most sharply and also according to G.N. Watson which 'shows Ramanujan at his best'.

A complete list of the identities on theta functions and related q-series stated by Ramanujan, is given in Chapter XVI of his Second Notebook published by the Tata Institute of Fundamental Research, Bombay, in 1957. Developing the theory of theta functions, Ramanujan also commences his study of elliptic functions. The first seventeen sections of the Sixteenth chapter are devoted primarily to q-series, while the latter twenty two sections, to a very thorough development of the theory of theta-functions. Altogether, there are 135 theorems, corollaries and examples, which constitute this chapter and form one of the most elegant illustration where the essential beauty and perfection of classical analysis is par excellence.

In 1985, in a Memoir published by the American Mathematical Society (vol. 53 No: 315), C.Adiga, B.C. Berndt, S. Bhargava and G.N. Watson have given a brief, yet very masterly survey of the contents of Chapter 16 and have supplied proofs for all the 135 identities on theta functions and q-series, given by Ramanujan. Although, Ramanujan's first efforts in this area are found in this Chapter, Yet there are much more deeper results on θ-functions, stated without proof, by him, found in his 'Lost' Notebook discovered by G. E. Andrews in 1976. Since these identities are capable of further generalisations in the context of the basic hypergeometric series and have ramifications in allied areas, their study has given a fresh impetus to the subject. The story of Ramanujan's work on θ-functions, here therefore, begins from these identities recorded in the 'Lost' Notebook.

As usual, throughout, we have adopted the following standard notations :

For any complex number a, and $|q| < 1$, we have

$(a)_k = (a; q)_k = (1-a)(1-aq)\ldots(1-aq^{k-1})$, $(a)_0 = 1$,

and $(a)_\infty = (a; q)\prod_0^\infty (1-aq^k)$.

The basic hypergeometric series ${}_{s+1}\phi_s$ is defined by

$${}_{s+1}\phi_s\left[\begin{matrix} a_1, a_2, \ldots a_{s+1}; \\ b_1, b_2, \ldots b_s \end{matrix} x\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_{s+1})_k x^k}{(b_1)_k (b_2)_k \ldots (b_s)_k (q)_k}$$

where $|x| < 1$ and the a's and b's are arbitrary, except that of course $(b_j)_k \neq 0$, $1 \leqslant j \leqslant s, 0 \leqslant k < \infty$.

We shall also, sometimes, abbreviate $(a_1;q)_k\,(a_2;q)_k \ldots (a_r;q)_k$ by $(a_1, a_2 \ldots a_r; q)_k$.

2. Partial Theta Function Identities

In 1976, G.E. Andrews [1] while on a visit to the Cambridge University, came across in the Library of the Trinity College a box donated to the Library, by the widow of (Late) Prof. G.N. Watson, containing more than one hundred papers, lying unscrutinised, in

Ramanujan's own distinctive handwriting, which had over six hundred formulae listed one after another, without proof. Andrews rightly termed these pages as Ramanujan's 'Lost' Notebook. Andrews gave proofs of most of the astounding identities contained therein. A section of the 'Lost' Notebook contains formulae for Mock-theta functions, False-theta functions and the theta functions.

We know that, the Jacobi's four theta functions are defined as

$$\theta_1(z;q) = 2q^{1/4} \sin z \prod_1^{\infty} (1-q^{2n}) \prod_1^{\infty} (1-2q^{2m}\cos 2z + q^{4m}), \quad (1)$$

$$\theta_2(z;q) = 2q^{1/4} \cos z \prod_1^{\infty} (1-q^{2n}) \prod_1^{\infty} (1+2q^{2m}\cos 2z + q^{4m}), \quad (2)$$

$$\theta_3(z;q) = \prod_1^{\infty} (1-q^{2n}) \prod_1^{\infty} (1+2q^{2m-1}\cos 2z + q^{4m-2}), \quad (3)$$

and

$$\theta_4(z;q) = \prod_1^{\infty} (1-q^{2n}) \prod_1^{\infty} (1-2q^{2m-1}\cos 2z + q^{4m-2}). \quad (4)$$

If we consider, for example, the function $\theta_1(z;q)$, then the product

$$\prod_1^{N} (1-q^{2n}\cos 2z + q^{4n}) \quad (5)$$

is a 'truncated' product of the infinite product involving z in the definition of $\theta_1(z;q)$. If we write

$$\theta_{1;N}(z;q) = 2q^{1/4} \sin z \prod_1^{\infty} (1-q^{2n}) \prod_{n=1}^{N} (1-q^{2m}\cos 2z + q^{4m}), \quad (6)$$

then

$$\lim_{N\to\infty} \theta_{1;N}(z;q) = \theta_1(z;q).$$

Ramanujan termed the trigonometric polynomials $\theta_{r;N}$; $r = 1,2,3,4$, as partial-theta functions.

By putting e=q and then transforming the resulting ${}_2\phi_1$ on the right of (2.16) by another well-known transformation of Sears [1], viz.,

$$(x)_\infty (e)_\infty \, {}_2\phi_1 \left[\begin{matrix} a,b;x \\ e \end{matrix} \right] = (b)_\infty \, (ax)_\infty \, {}_2\phi_1 \, [e/b,x; \, ax; b].$$

Agarwal obtained a transformation which is the main result (2.14) of Andrews, after renaming the variables.

This further suggested that a systematic study of all the known three and four term relations of Sears may give a variety of identities of the type (2.14).

Indeed, A.K. Agarwal in his doctoral thesis in 1985 [1] made a systematic study of all the three and four term relations between ${}_3\phi_2$ series of Sears [1] and showed that there do exist, at least, three more identities of the type (2.14) which are essentially different from it.

However, (2.15) appeared, till very recently to have evaded unification in terms of the general basic hypergeometric transformation theory. A. Verma and V.K. Jain [1], a few weeks back have succeeded in showing that even (2.15) does belong to a more general class of basic hypergeometric identities. In their still unpublished manuscript, communicated to me, they have shown that it could be deduced from the general transformation.

$$\begin{aligned} {}_3\phi_2 \left[\begin{matrix} a,c,d; \dfrac{aqx}{cd} \\ aq/c, aq/d \end{matrix} \right] &= \frac{(ax;q)_\infty}{(x;q)_\infty} \, {}_5\phi_4 \left[\begin{matrix} \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, \; aq/cd; q \\ aq/c, \; aq/d, \; ax, \; q/x \end{matrix} \right] \\ &+ \frac{(a,aq/cd,axq/c,axq/d; \; q)_\infty}{(aq/c,aq/d,aqx/cd,1/x; \; q)_\infty} \, {}_5\phi_4 \\ &\left[\begin{matrix} x\sqrt{a}, -x\sqrt{a}, x\sqrt{aq}, -x\sqrt{aq}, axq/cd; q \\ axq/c, \; axq/d, \; xq, ax^2 \end{matrix} \right]. \end{aligned} \tag{17}$$

In fact, (2.17) is a known quadratic transformation due to Gasper and Rahman [1] who deduced it by letting $n \to \infty$ in the Bailey's transformation [1] between terminating nearlypoised ${}_5\phi_4$ and terminating well-poised ${}_{12}\phi_{11}$. Varma and Jain gave several other transformations similar to (2.17) to deduce a huge variety of new partial-theta function identities.

$$T_n(4,4;q,q^2)=\frac{q^{n^2}\prod_{m=0}^{\infty}(1+q^{2m+1})}{(1-q)(1-q^3)\ldots(1-q^{2n-1})(1-q^4)(1-q^8)\ldots(1-q^{4n})}. \quad (3)$$

From these, Ramanujan derived a large number of q-series summations. Andrews (3) has shown that Ramanujan's expanssions have some elegant partition theoretic inter-pretations also.

Andrews (3) deduced Ramanujan's results by proving his Lemma I in the form :

$$\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n\,(-q/a;q^2)_n\,(b;q^2)_n\,(c;q^2)_n\,(q^2/bc)^n}{(q^2;q^2)_{2n}}$$

$$=\frac{(q^2/b;q^2)_\infty(q^2/c;q^2)_\infty}{(q^2;q^2)_\infty\,(q^2/bc;q^2)_\infty}\sum_{N=-\infty}^{\infty}\frac{q^{N^2+2N}\,(a/bc)^N\,(b;q^2)_N\,(c;q^2)_N}{(q^2/b;q^2)_N\,(q^2/c;q^2)_N}. \quad (4)$$

One can show that Lemma I of Andrews can be easily extended in a more general basic hypergeometric setting and thus giving the possibility of studying more general expasions of the type (3.1).

In the sequel, we also get certain expansions involving false theta functions. The two false theta functions were originally defined and studied by L.J. Rogers [1]. They are simply theta series with the 'wrong' signs, i.e.,

$$\theta_{1,f}(z;q)=2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z$$

$$\theta_{2,f}(z;q)=2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\cos(2n+1)z.$$

In fact, Bhaskar Srivastava [1] proved that

$$\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n\,(-q/a;q^2)_n\,(a_r);q^2)_n\,(tq^\lambda)^n}{(q^2;q^2)_{2n}\,((\beta_s);q^2)_n}$$

To derive (2.15), Varma and Jain proved the simple quadratic transformation

$$(-aq)_\infty\ (q)_\infty\ {}_3\phi_2\left[\begin{matrix}a\sqrt{q},\ -a\sqrt{q},\ aq;\ q\\ a^2q,\ 0\end{matrix}\right]$$

$$=\sum_0^\infty\ (1+aq^{2n+1})\,(-a^3)^n\ q^{3n^2+2n}\ . \qquad (18)$$

Setting $a=q$, $x=-a$ and letting $c, d \to\infty$ in (2.17) they obtained

$$\sum_0^\infty\ \frac{(q;q)_{2n}}{(q,-aq,-q/a;q)_n}=(1+a)\sum_0^\infty\ (-a)^n\ q^{n^2+n}\ -a$$

$$\frac{(q;q)_\infty}{(-q/a;q)_\infty}\times$$

$$\times\ {}_3\phi_2\left[\begin{matrix}a\sqrt{q},\ -a\sqrt{q},\ aq;\ q\\ a^2q,0\end{matrix}\right]. \qquad (19)$$

Transforming the ${}_3\phi_2$ on the right of (2.19) by (2.18), we get (2.15).

From (2.14) and (2.15), by specialising the parameters Andrews [2] deduced all the partial-theta function identities stated by Ramanujan and also gave partition theoretic interpretations of them.

3. Partial Theta Function Expansions

Still another aspect of the θ-functions studied by Ramanujan is the expansions of partial theta functions of the type

$$\sum_{N=0}^\infty\ T_N(i,j;q,q_1)\,\theta_{i,N}\ (z;q)=\theta_j\,(z;q_1).$$

$$i,j=1,2,3,4. \qquad (1)$$

Now the question is, if such expansion exists, what is the form of these coefficients $T_N\ (i, j; q, q_1)$? Ramanujan's wonderful observation was that in a number of cases these coefficients have very elegant closed forms, two of which we give below :

$$T_n(4,4;q,q^3)=\frac{q^{2n^2}}{(1-q^2)\,(1-q^4)\ldots(1-q^{4n})} \qquad (2)$$

$$\times\{1-\frac{c}{bq}-\frac{c}{cq}+\frac{c}{q}+\frac{de\,(1-b)\,(1-c)}{bcq\,(1-d)}\},$$

to sum the n-series on the righthand side of (3.5) by putting

$$\alpha_1=b,\ \alpha_2=c,\ \alpha_3=dq^2,\ \beta_j=d,\ e=q^2,\ \lambda=2 \text{ and } t=\frac{1}{bcq^2},$$

we get, after some simplification,

$$\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n\,(-q/a;q^2)_n(b;q^2)_n(c;q^2)_n\,(dq^2;q^2)_n\,(\frac{1}{bc})^n}{(q^2;q^2)_{2n}\,(d;q^2)_n}$$

$$=\prod\left[\begin{matrix}q^2/b,\ q^2/c\\ q^2,\ 1/bc\end{matrix};q^2\right]\left[\{2-1/b-1/c+\frac{d\,(1-b)\,(1-c)}{bc\,(1-d)}\}+\right.$$

$$\sum_{N=1}^{\infty}\frac{(a^N+a^{-N})q^{N^2}\ (b;q^2)_N\ (c;q^2)_N\ \ (dq^2;q^2)_N\,(\frac{1}{bc})^N}{(d;q^2{}_N\,(q^2/b;q^2)_N\,(q^2/c;q^2)_N}\times$$

$$\left.\times\{1-(\frac{1}{b}+\frac{1}{c})\,q^{2N}\ +q^{4N}\ +dq^{2N}\ \frac{(1-bq^{2N})\ (1-cq^{2N})}{bc(1-dq^{2N})}\}\right]. \tag{1}$$

Letting $d\to\infty$ in (4.1), we get Andrews' Lemma (3.4).

Putting $a=-e^{2iz}$ and letting $b\to\infty$, $C\to\infty$ in (4.1) with d replaced by—d, we get after some simplification

$$\sum_{n=0}^{\infty}\ \frac{q^{2n^2-2n}\ (1+dq^{2n})}{(q^2;q^2)_{2n}}\theta_{4;n}\ (z;q)=(d-2q^{-3}\cos 2z)\ \theta_4\ (z;q^3). \tag{2}$$

Dividing by d and then taking limit as $d\to\infty$, we have

$$\sum_{n=0}^{\infty}\ \frac{q^{2n^2}}{(q^2;q^2)_{2n}}\theta_{4;n}\ (z;q)=\theta_{4,n}\ (z;q^3)\,, \tag{3}$$

which is precisely Ramanujan's expansion (Andrews [3], eqn, (1.10).

In the identity (4.3), writing $\pi/3$ for z, we get after some simplification

$$= {}_r\phi_{s+1}\left[\begin{matrix}(a_r) \\ q^2, (\beta_s);\end{matrix}\ ;\ q^2; tq\right] + \sum_{N=1}^{\infty} \frac{(a^N + a^{-N})\, q^{N^2}\ ((a_r); q^2)_N\ (tq^\lambda)^N}{(q^2;q^2)_{2N}\ ((\beta_s);\ q^2)_N} \times$$

$$\times \sum_{n=0}^{\infty} \frac{((a_r)q^{2N};q^2)_n\, (tq^\lambda)^n}{((\beta_s)q^{2N};q^2)_n\, (q^2;q^2)_n\, (q^{4N+2}, q^2)}, \qquad (5)$$

where (a_r) denotes the sequence $a_1, a_2, \ldots, a_r$ and λ is a suitable constant.

Proof :

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n\, (-q/a;q^2)_n\, ((a_r);q^2)_n\, (tq^\lambda)^n}{(q^2;q^2)_{2n}\ ((\beta_s);\ q^2)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2+\lambda n}\ a^{-n}\ t^n\ (-aq^{-2n+1};q^2)_{2n}\ ((a_r)\ ;q^2)_n}{(q^2;q^2)_{2n}\ ((\beta_s);\ q^2)_n}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^{-n+m}\ q^{n^2-2nm+m^2+\lambda n}\ ((a_r);\ q^2)_n t^n}{(q^2;q^2)_m\ (q^2;q^2)_{2n-m}\ ((\beta_s);\ q^2)_n}$$

$$= \sum_{N=-\infty}^{\infty} a^N q^{N^2} \sum_{n=0}^{\infty} \frac{((a_r);\ q^2)_n\, (tq^\lambda)^n}{(q^2;q^2)_{n+N}\ (q^2;q^2)_{n-N}\ ((\beta_s);\ q^2)_n}$$

$$= {}_r\phi_{s+1}\left[\begin{matrix}(a_r); \\ q^2, (\beta_s);\end{matrix}\ q^2;\ tq^\lambda\right] +$$

$$+ \sum_{N=1}^{\infty} (a^N + a^{-N}) q^{N^2} \sum_{n=0}^{\infty} \frac{((a_r);q^2)_{n+N}\ (tq^\lambda)^{n+N}}{(q^2;q^2)_n\ (q^2;q^2)_{n+2N}((\beta_s);q^2)_{n+N}},$$

which on simplification gives the desired result (3.5).

4. Applications of Theorem (3 5)

Using the following known summation formula due to Andrews [6],

$${}_3\phi_2\left[\begin{matrix}b, c, dq \\ d, e\end{matrix}\ ;\ q\ ;\frac{e}{bcq}\right] = \prod\left[\begin{matrix}e/b, & e/c \\ e/bcq, & e\end{matrix}\ ;\ q\right]$$

$$(q)_\infty \sum_0^\infty \frac{q^{2N^2}\,(q^3;q^6)_n}{(q;q^2)_n\,(q^2;q^2)_n}$$

$$\prod_1^\infty (1-q^{18n-9})\sum_{-\infty}^{\infty}(-1)^n\, q^{3n^2-2n}\,. \qquad (4)$$

This identity (4.4) looks apparently insignificant, but it has far reaching implications in the theory of partitions. Andrews [3] has shown that the above identity is equivalent to the following partition-theoretic result :

Let P_1 (n) denote the number of partition of n subject to the following three conditions : (i) no part appears twice, (ii) no odd part exceeds the numbr of even parts, (iii) among the even parts (arranged in non-increasing size) only the second, fourth, sixth etc. may be subsequently repeated. Let P_2 (n) denote the number of partitions of n into parts $\in \pm 2, \pm 3, \pm 4, \pm 8 \pmod{18}$. Then $P_1(n) = P_2(n)$, for all n.

As an example of the above theorem, $P_1(10)=7$, since the partitions as per the conditions of the theorem are 10, 8+2, 8+1+1, 6+4, 6+2+2, 6+2+1+1, 4+2+2+1+1, wherea $P_2(10)=7$ since the partitions are 10, 8+2, 4+4+2, 4+3+3, 4+2+2+2, 3+3+2+2, 2+2+2+2+2.

Putting d=0, in (4.2), we have

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-2n}}{(q^2;q^2)_{2n}}\,\theta_{4;n}(z;q) = -2q^{-3}\cos 2z\;\theta_4(z;q^3). \qquad (5)$$

Next, putting $a=e^{2iz}$ and letting $b\to\infty$, $c\to\infty$ in (4.1) we get

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-2n}\,(1-dq^{2n})}{(q^2;q^2)_{2n}}\,\theta_{3;n}(z;q) = (2q^{-3}\cos 2z - d)\,\theta_3(z;q^3). \qquad (6)$$

Dividing by d and then taking the limit as $d\to\infty$, we have

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_{2n}}\,\theta_{3;n}(z;q) = \theta_3(z;q^3). \qquad (7)$$

Again putting $a=-e^{2iz}$, $b=-q$ and letting $c=\infty$, in (4.1), we get

$$\frac{1}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-2n}\,(-q;q^2)_n\,(1-dq^{2n})}{(q^2;q^2)_{2n}}\,\theta_{4;n}\,(z;q)$$

$$=(d-\frac{1}{q}+2q^{-2}\cos 2z)\,\theta_4\,(z;q^2). \qquad (8)$$

Dividing by d and then taking the limit as $d\to\infty$, we get

$$\frac{1}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}\,(-q;q^2)_n}{(q^2;q^2)_{2n}}\,\theta_{4;n}\,(z;q)=\theta_4\,(z;q^2), \qquad (9)$$

which is again Ramanujan's expansion (Andrews (3), eq. 1. 11)).

Still another interesting particular case is obtained by putting $a=e^{2iz}$, $b=-q$ and letting $c\to\infty$, in (4.1)
we get

$$\frac{1}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-2n}\,(-q;q^2)_n\,(1-dq^{2n})}{(q^2;q^2)_{2n}}\,\theta_{3;n}\,(z;q)$$

$$=-(d-\frac{1}{q}-2q^{-2}\cos 2z)\,\theta_2\,(z;q^2).$$

Dividing by d and then taking the limit as $d\to\infty$, we get

$$\frac{1}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}\,(-q;q^2)_n}{(q^2;q^2)_{2n}}\,\theta_{3;n}\,(z;q)=\theta_3\,(z;q^2). \qquad (11)$$

The particular cases (4.2) to (4.11) give expansions of a $\theta_{i,N}\,(z,q)$ function in terms of another $\theta_i\,(z;q)$ function.

5. Some More Divergent Special Cases of (4.1)

Putting $a=e^{2iz}$, $b=-q$, $c=q$, in (4.1), we get,

$$\frac{(-q^{-2};q^2)_\infty}{(-q;q^2)_\infty\,(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n\,q^{-2n}\,(-q;q^2)_n(q;q^2)_n(1-dq^{2n})}{(q^2;q^2)_{2n}}$$

$$\theta_{3,n}\,(z;q)$$

$$=-(d+\frac{d}{q^2}+\frac{2}{q}\cos 2z)\,\theta_4\,(z;q). \qquad (1)$$

Dividing by d and then taking the limit as $d\to\infty$, we get on simplification

$$\times \sum_{n=0}^{\infty} \frac{((a_r)\, q^N)_n\, (tq^\lambda)^n}{(q)_n\, (q^{2N+2})_n\, ((\beta_s)^N)^n}. \qquad (1)$$

Proof :

$$\sum_{n=0}^{\infty} \frac{(-a)_{n+1}\, \{-q/a\}_n\, ((a_r))_n\, (tq^\lambda)^n}{(q)_{2n+1}\, ((\beta_s))_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{\frac{n^2+n}{2}+\lambda n} a^{-n}\, (-aq^{-n})_{2n+1}\, ((a_r))_n\, t^n}{(q)_{2n+1}\, ((\beta_s))_n}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^{-n+m}\, q^{\frac{n^2+n}{2}+\lambda n} q^{-nm+\frac{m^2-m}{2}}\, ((a_r))_n\, t^n}{(q)_m\, (q)_{2n+1-m}\, ((\beta_s))_n}$$

$$= \sum_{N=-\infty}^{\infty} a^N q^{\frac{N^2-N}{2}} \sum_{n=0}^{\infty} \frac{((a_r))_n\, ((tq^\lambda)^n}{(q)_{n+N} (q)_{n+1-N}\, ((\beta_s))_n}$$

$$= \sum_{N=1}^{\infty} (a^N + a^{1-N})\, q^{\frac{N^2-N}{2}}$$

$$\sum_{n=0}^{\infty} \frac{((a_r))_{n+N-1}\, (tq^\lambda)^{n+N-1}}{(q)_n\, (q)_{n+2N-1}\, ((\beta_s))_{n+N-1}}$$

$$= \sum_{N=1}^{\infty} (a^N + a^{1-N})\, q^{\frac{N^2-N}{2}} \frac{((a_r))\, q^{N-1})_n (tq^\lambda)^{N-1}}{(q)_{2N-1}\, ((\beta_s) q^{N-1})_n}$$

$$\times \sum_{n=0}^{\infty} \frac{((a_r)\, q^{N-1})_n\, (tq^\lambda)^n}{(q)_n\, (q^{2N})_n\, ((\beta_s) q^{N-1})_n}$$

$$= \frac{(1+a)}{(1-q)}\, {}_r\phi_{s+1} \left[\begin{matrix} (a_r) \\ (\beta_s),\ q^2 \end{matrix} ;\ tq^\lambda \right] +$$

$$\frac{2\,(-q^2;q^2)_\infty}{(-q;q^2)_\infty\ (q;q^2)_\infty}\ \sum_{n=0}^{\infty}\ \frac{(-1)^n}{(q^4;q^4)_n}\,\theta_{3;n}\ (z;q)=\theta_4\ (z;q). \tag{2}$$

This is precisely the divergent expansion $(1.12)_R$ of Ramanujan proved by Andrews (3). This has to be interpreted as in Andrews (3).

Next, putting $a=-e^{2iz}$, $b=-q$, $c=q$, in (4.1), we get

$$\frac{(-q^{-2};b^2)_\infty}{(-q;q^2)_\infty\ (-q;q^2)_\infty}\ \sum_{n=0}^{\infty}\ \frac{(-1)^n\,q^{-2n}\,(-q;q^2)_n\,(q;q^2)_n(1-dq^{2n})}{(q^2;q^2)_{2n}}\ \theta_{4;n}\ (z;q)$$

$$=-\left(d+\frac{d}{q^2}-\frac{2}{q}\cos 2z\right)\theta_3\ (z;q). \tag{3}$$

This, again is a divergent expansion and has to be interpreted as in Andrews (3) Dividing by d and then taking the limit as $d\to\infty$, we get

$$\frac{2\,(-q^2;q^2)_\infty}{(-q;q^2)_\infty\ (-q;q^2)_\infty}\ \sum_{n=0}^{\infty}\ \frac{(-1)^n}{(q^4;q^4)_n}\,\theta_{4;n}\ (z;q)=\theta_3\ (z;q). \tag{4}$$

The special cases mentioned in §5 are a sort of divergent reciprocal relations between $\theta_{4;n}\ (z,q)$ and $\theta_3\ (z,q)$.

6. Another Expansion Theorem

We find that the expansion theorem (3.5) does not yield expansion for $\theta_{1;n}$ and $\theta_{2;n}$ series. To obtain such expansions we give still another expansion theorem [Bhasker Srivastava (1)], namely, subject to suitable conditions of convergence,

$$\sum_{n=0}^{\infty}\ \frac{(-a)_{n+1}\,(-q/a)_n\,((a_r))_n\,(tq^\lambda)^n}{(q)_{2n+1}\,((\beta_s))_n}$$

$$=\frac{(1+a)}{(1-q)}\ {}_{r}\phi_{s+1}\left[\begin{matrix}(a_r)\\(\beta_s),\ q^2\end{matrix};\ tq^\lambda\right]$$

$$+\ \sum_{N=1}^{\infty}\ \frac{(a^{1+N}+a^{-N})\,q^{\frac{N^2+N}{2}}\,((a_r))_N\,(tq^\lambda)^N}{(q)_{2N+1}\ ((\beta_s))_N}$$

$$=(1-d)\,\theta_2(z;q^2)+2q^{1/2}\cos z\,\theta_3(z;q^2). \tag{2}$$

Putting $d=1$ in (7.2), we have

$$\frac{q^{1/4}}{(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}(-q^2;q^2)_n}{(q^4;q^2)_{2n}}\theta_{2;n}(z;q)$$

$$=2q^{1/2}\cos z\,\theta_3(z;q^2). \tag{3}$$

Putting $d=0$, we have

$$\frac{q^{1/4}}{2(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}(-q^2;q^2)_n}{(q^2;q^2)_{2n+1}}\theta_{2;n}(z;q)$$

$$=\theta_2(z;q^2)+2q^{1/2}\cos z\,\theta_3(z;q^2). \tag{4}$$

Finally dividing by d and then letting $d\to\infty$, we get

$$\frac{q^{1/4}}{(-q^2,q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q^2;q^2)_n}{(q^2;q^2)_{2n+1}}\theta_{2;n}(z;q)=\theta_2(z;q^2) \tag{5}$$

Next putting $a=e^{2iz}$, $b=q^2$ and letting $c\to\infty$ in (7.1), we have

$$\frac{1}{2q^{1/4}(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2-n}(1-dq^{2n}}{(q^{2n+2},q^2)_{n+1}}\theta_{2;n}(z;q)$$

$$=\cos z\,(1+d)\,q^{-1/2}\theta_{2,f}(z;q^2)-2\sin z\sum_{N=1}^{\infty}(-1)^N q^{2N^2}\sin 2Nz. \tag{6}$$

Dividing by d and then letting $d\to\infty$ in (7.6), we get

$$\frac{q^{1/4}}{2(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{(q^{2n+2};q^2)_{n+1}}\theta_{2;n}(z;q)=\theta_{2,f}(z;q^2), \tag{7}$$

which is precisely Ramanujan's evpansion (5.3) (Andrews 3; $(5.3)_R$).

Further, putting $a=-e^{2iz}$, $b=q^2$ and letting $c\to\infty$ in (7.1), we have

$$\frac{1}{2q^{1/4}(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2-n}(1-dq^{2n}}{(q^{2n+2};q^2)_{n+1}}\theta_{1;n}(z;q)$$

$$+\sum_{N=1}^{\infty}\frac{(a^{1+N}+a^{-N})q^{\frac{N^2+N}{2}}((a_r))_N(tq^{\lambda})^N}{(q)_{2N+1}((\beta_s))_N}$$

$$\sum_{n=0}^{\infty}\frac{((a_r))\,q^N)_n(tq^{\lambda})^n}{(q)_n(q^{2N+2})_n((\beta_s)\ q^N)_n}\ .$$

This gives the desired result (6.1).

(6.1) is a generalisation of Lemma 2 of Andrews [3; (5.1)] which is obtained by taking $r=2$, $s=0$, $\lambda=2$ $t=\frac{1}{a_1a_2}$ in (6.1).

7. Applications of (6.1)

Using the summation formula of §4, by taking $a_1=b$, $a_2=c$, $a_3=dq$, $\beta_1=d$, $c=q^2$, $t=\frac{1}{cbq}$, we get, on some simplification and finally replacing q by q^2 throughout,

$$\sum_{n=0}^{\infty}\frac{(-a;q^2)_{n+1}(-q^2/a;q^2)_n(b;q^2)_n(c;q^2)_n(dq^2;q^2)_n(q^2/bc)^n}{(q^2;q^2)_{2n+1}(d;q^2)_n}$$

$$=(1+a)\prod\left[\begin{matrix}q^4/b,\ q^4/c\\ q^2,\ q^2/bc\end{matrix};q^2\right]\left\{1-\frac{q^2}{b}-\frac{q^2}{c}+q^2+\frac{dq^2(1-b)(1-c)}{bc(1-d)}\right\}+$$

$$+\sum_{N=1}^{\infty}\frac{(a^{1+N}+a^{-N})q^{N^2+N}(b;q^2)_N(c;q^2)_N(dq^2;q^2)_N(q^2/bc)^N}{(d;q^2)_N(q^4/b;q^2)_N(q^4/c;q^2)_N}$$

$$\times\prod\left[\begin{matrix}q^4/b,\ q^4/c\\ q^2,\ q^2/bc\end{matrix};q^2\right]\left\{1-\frac{q^{2N+2}}{b}-\frac{q^{2N+2}}{c}+q^{4N+2}+\right.$$

$$\left.+\frac{dq^{2N+2}(1-bq^{2N})(1-cq^{2N})}{bc(1-dq^{2N})}\right\}. \qquad (1)$$

Putting $a=e^{2iz}$, $b\to\infty$, $c=-q^2$, in (7.1), we have

$$\frac{q^{1/4}}{(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}(-q^2;q^2)_n(1-dq^{2n})}{(q^2;q^2)_{2n+1}}\theta_{2;n}(z;q)$$

$$=\sin z - q^{-1/2}\ (1+d)\ \theta_{1;f}\ (z;q^2)+2\cos z \sum_{N=1}^{\infty} q^{2N^2} \sin 2Nz, \tag{8}$$

Dividing by d and then letting $d\to\infty$ in (7.8), we have

$$\frac{q^{1/4}}{2\,(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n\ q^{n^2+n}}{(q^{2n+2};\ q^2)_{n+1}} \theta_{1;n}\ (z;q)=\theta_{1,f}\ (z;q^2). \tag{9}$$

Lastly, putting $a=-e^{2iz}$ and letting $b\to\infty$ and $c\to\infty$ in (7.1), we have

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}\ (1-dq^{2n})}{(q^2;q^2)_{2n+1}} \theta_{1;n}\ (z,q)$$

$$=-(dq^{-1/2}+2q^{-5/2}\cos 2z)\ \theta_1\ (z;q^3). \tag{10}$$

Dividing by d, then letting $d\to\infty$, we get

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_{2n+1}} \theta_{1;n}\ (z;q)=-q^{-1/2}\ \theta_1\ (z;q^3). \tag{11}$$

Similar substitutions yield not only the identities given by Ramanujan (and proved by Andrews [3]) but as we see above, numerous new identities, not stated by Ramanujan. Our object has been only to indicate the possibility of generalising Ramanujan's results in the basic hypergeometric setting, the theory of which has been extensively studied.

I do hope that the present article will help research workers in this area to see the vast potentialities of extensions, in terms of known standard functions, which may serve to obtain new results of interest in number theory.

REFERENCES

Agarwal, R.P., "A family of basic hypergeometric and combinatorial identities and certain summation formulae". *J. Pure & Applied Maths.* 12 (1981).

———, "On the paper "Ramanujan's Lost Notebook-I Partial theta functions". *Adances in Maths* (1984).

———, "Ramanujan and basic hypergeometric series". *Current Science 52 (1983).*

Agarwal A.K. "A study of partial—theta functions and generalization of certain polynomials of Exton". *Ph.D. Thesis approved by the University of Lucknow* (1985).

Andrews, G.E., "An introduction to Ramanujan's 'Lost' Notebook" American Math. Monthly 86 (1979), 89-108.

———, "Ramanujan's "Lost" Notebook-I", Partial theta functions", *Advances in Maths.* 41 (1981), 137-172.

———, "Ramanujan's "Lost" Notebook-II-function expansions". *Ibid* 41 (1981).

———, "Ramanujan's 'Lost' Notebook-III-The Rogers—Ramanujan Continued fraction". *Ibid* 41 (1981), 186-208.

———, "Ramanujan's "Lost" Notebook-IV-Stacks and Alternating parity in Partitions". *Ibid* 53 (1984), 55-74.

———, On Q-analogues of Watson's and Whipple's summation's", *SIAM J. Math. Anal.* 7 (1976).

Bailey, W.N., "A transformation of nearly-poiseed basic hypergeometric series" *J. London Math. Soc.* 22 (1947), 237-240.

C. Adiga, B.C. Berdnt et. al. "Chapter 16 of Ramanujan's Second Notebook : Theta functions and Q-series", *Memories of American Math. Soc.*, 53 No. 315 (1985).

Gasper, G. and Rehman, M., "Positivity of the Poisson kernel for the continuous q-Jacobi polynomials and some quadratic transformations for basic hypergeometric series" (to appear).

Rogers, L.J., "On two theorems of combinatory analysis and some allied identities", *Proc. London Math. Soc.* (2) 16 (1917), 315-336.

Ramanujan, S., "Notebooks (2 Vols.)", *Tata Institute of Fundamental Research, Bombay* (1957).

Errata

page	line	for	read
3	3	region	regime
	1st line last para	struck	stuck
	3rd from the bottom	man	chain
12	5	operation	operations
17	1	$\left\{1+\frac{2}{3^3-3}+\cdots\right\}$	$\Gamma\left\{1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots+\frac{2}{(3n)^2-3n}\right\}$
11	3rd from the bottom	$\frac{1}{a_1}+\frac{1}{[a_2}+\cdots$	$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}=\frac{2n}{a_1+a_n}$
24	last	gived	given
28	2nd from the bottom	$\frac{1}{A_{2r}}$	$\frac{1}{A_{2n}}$
29	1–3	2 log 3 tan $4\frac{e^x-1}{e^x+1}$	2. log 3. tan $4.\frac{e^x-1}{e^x+1}$
32	4	plaine	plain

Sears, D.B., "On the transformation theory of basic hypergeometric series". *Proc. London Math. Soc.* (2) 53 (1951), 158-180.

———, "Transformations of basic hypergeometric function of special type". *Proc. London Math. Soc.* (2) 53 (1951), 467-483.

Srivastava, Bhaskar, "Certain basic hypergeometric identities and continued fractions associated with Ramanujan". *Ph.D. Thesis approved by the University of Lucknow* (1986).

Varma, A. and Jain, V.K., "Some quadratic transformation of basic hypergeometric series and identities in Ramanujan's "Lost" Notebook (unpublished manuscript).

33	9th from the bottom	Rome	like
38	22	Naster	Master
43	last	$\phi(k)\ (-x)^k/k$	$\phi(k)\ (-x)^k/k$!
44	5	a(–) circular	a circular
47	1st paragraph last line	Neille	Neville
48	6	Hary	Hardy
49	7	Though	Through
50	1st line last para	his	His
52	3	sent	send
57	19	eclusively	exclusively
58	3	Show	Snow
73	2	weaks	weeks
78	8th from the bottom	pilling	piling
82	3rd line last para	entitte	entities
85	last line	$\frac{\text{Numerator}}{[(1-x)(1-x^2)(1-x)^3....]^6}$	$\frac{\text{Numerator}}{[(1-x)(1-x^2)(1-x^3)....]^6}$
86	5–7	general Conjecture ....for every m,	general conjecture that if $\delta = 5^a, 7^b, 11^e$ and $24\lambda \equiv 1 \pmod{\delta}$ then $p(m\delta + \lambda) \equiv 0 \pmod{\delta}$ for every m,
94	4	Macbeth	Lady Macbeth
	11	lusture	luster
101	6	as had	as he had
103	9	India	Indian

Errata

page	line	for	read
3	3	region	regime
	1st line last para	struck	stuck
	3rd from the bottom	man	chain
12	5	operation	operations
17	1	$\left\{1+\frac{2}{3^3-3}+\cdots\right\}$	$\Gamma\left\{1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots+\frac{2}{(3n)^2-3n}\right\}$
11	3rd from the bottom	$\frac{1}{a_1}+\frac{1}{[a_2}+\cdots$	$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}=\frac{2n}{a_1+a_n}$
24	last	gived	given
28	2nd from the bottom	$\frac{1}{A_{2r}}$	$\frac{1}{A_{2n}}$
29	1–3	2 log 3 tan	2. log 3. tan
		$4\frac{e^x-1}{e^x+1}$	$4.\frac{e^x-1}{e^x+1}$
32	4	plaine	plain

33	9th from the bottom	Rome	like
38	22	Naster	Master
43	last	$\phi(k)\ (-x)^k/k$	$\phi(k)\ (-x)^k/k$!
44	5	a(–) circular	a circular
47	1st paragraph last line	Neille	Neville
48	6	Hary	Hardy
49	7	Though	Through
50	1st line last para	his	His
52	3	sent	send
57	19	eclusively	exclusively
58	3	Show	Snow
73	2	weaks	weeks
78	8th from the bottom	pilling	piling
82	3rd line last para	entitte	entities
85	last line	$\frac{\text{Numerator}}{[(1-x)(1-x^2)(1-x)^3....]^6}$	$\frac{\text{Numerator}}{[(1-x)(1-x^2)(1-x^3)....]^6}$
86	5–7	general Conjecture ....for every m,	general conjecture that if $\delta = 5^a, 7^b, 11^e$ and $24\lambda \equiv 1 \pmod{\delta}$ then $p(m\delta + \lambda) \equiv 0 \pmod{\delta}$ for every m,
94	4	Macbeth	Lady Macbeth
	11	lusture	luster
101	6	as had	as he had
103	9	India	Indian

113	13	found Bhaskar	found in Bhaskar
116	6	Waston	Watson
124	5	Wiltkinson	Willkinson
126	7	interesting	intersecting
128	last	$\left(4^3\sqrt{\frac{2}{3}}5_3\sqrt{\frac{1}{3}}\right)^{\frac{1}{6}}$	$\left(4^3\sqrt{\frac{2}{3}}-5\,{}_3\sqrt{\frac{1}{3}}\right)^{\frac{1}{6}}$
130	11	$=0\,9$	$=0$
	2nd from bottom	$B_{\frac{1}{4}n+1}$	$B_{\frac{1}{2}n+1}$
134	5	; in the numerator,	
	8	and denominator	!
139	7	1637	6137
	5th from bottom	$\ldots\left(1-\frac{1}{5^n}\right)$	$\ldots\left(1-\frac{1}{5^n}\right)\ldots$
	3rd from bottom	$\int_0^\infty \frac{x^{n-1}}{e^{2nx}-1}dx$	$\int_0^\infty \frac{x^{n-1}}{e^{2\pi x}-1}dx$
	last	$\frac{(n-1)}{(2\pi)^n}$	$\frac{(n-1)\,!}{(2\pi)^n}$
141	12	$\frac{\ldots.1\ldots.}{6(x^2-x)}$	$\frac{\ldots.1\ldots.}{6(x^3-x)}$
	13	$+\frac{529}{x^{21}}$	$-\frac{529}{x^{21}}$
146	last	$\frac{1}{3^n}(-1)^{\frac{1}{6}(n-\ )}$	$\frac{1}{3}n\,(-1)^{\frac{1}{6}(n-4)}$
		or : $\frac{1}{6^n}$	$\frac{1}{6}n$
150	last	71, 151, 231	7 !, 15 !, 23 !
		41, 121, 201	4 !, 12 !, 20 !
		91, 171, 251	9 !, 17 !, 25 !
		41, 121, 201	4 !, 12 !, 20 !

151	last but 1	$\int_0^1 \frac{x^p}{\sqrt{1-x^2}}\,dx$	$=\int_0^1 \frac{x^p}{\sqrt{1-x^2}}\,dx$
152	6th from the bottom	$+\frac{1}{2}+\frac{1}{3^{n+1}}+\ldots$	$+\frac{1}{2}\cdot\frac{1}{3^{n+1}}\neq\ldots$
	last	σ_2	σ_2
154	(ii)–(iv)	$\left\{(\frac{1}{4})\right\}^2$	$\left\{\Gamma(\frac{1}{4})\right\}^2$
157	4	11594	11595
165	6	$\left(1-\frac{4}{15^2}\right)\ldots\frac{1}{21^5}\Big)^{-7}$	$\left(1-\frac{4}{15^2}\right)^5\left(1-\frac{1}{21^2}\right)^{-7}$
	7th from the bottom	$\tan^{-1}e^{-\frac{1}{2}\pi x}$	$\tan^{-1}e^{-\frac{1}{2}\pi x}$
166	last	$e^{\frac{8}{\Pi}\phi(1)}$	$=e^{\frac{8}{\Pi}\phi(i)}$
171	4	$a\geqslant 0$	$a_2\geqslant 0$
175	8	$=\frac{1}{9}$	$=\frac{1}{96}$
176	11	$\frac{9}{6}$	$\frac{2}{9}$
	13	$\frac{1}{\sqrt{2n}}$	$\frac{1}{\sqrt{2n}}\Big)$
184	last but 1	2p.230	2p.226
187	7	imginary	imaginary
188	2	$\overline{(2n)}$	$\overline{(2n)}\,!$
	last but 1	$q=e^{-\pi}$	$q=e^{-\pi}$
189	6	μ 190	μ. 186
196	10	If q=	If p=
204	19	$\lambda(m^2)$	$\lambda(z^2)$
205	5th from the bottom	p.393	p.290

217 6th from the bottom implict → implicit

219 4

$$\sum_{n>0} \frac{q^{n^2+n\infty}}{[q;q]_n} \prod_{n\neq 0,\pm 1 (\mathrm{mods})} (1-q^n)^{-1}$$

$$\sum_{n>0} \frac{q^{n^2+n}}{[q;q]_n} = \prod_{n\neq 0,\pm 1 (\mathrm{mods})} (1-q^n)^{-1}$$